Time dependent behavior of materials and structures

Learning Resource 1 – Theoretical Frame of Sustainable Development

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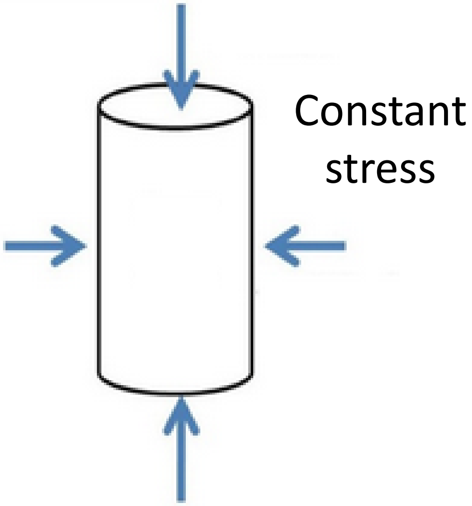
Outline

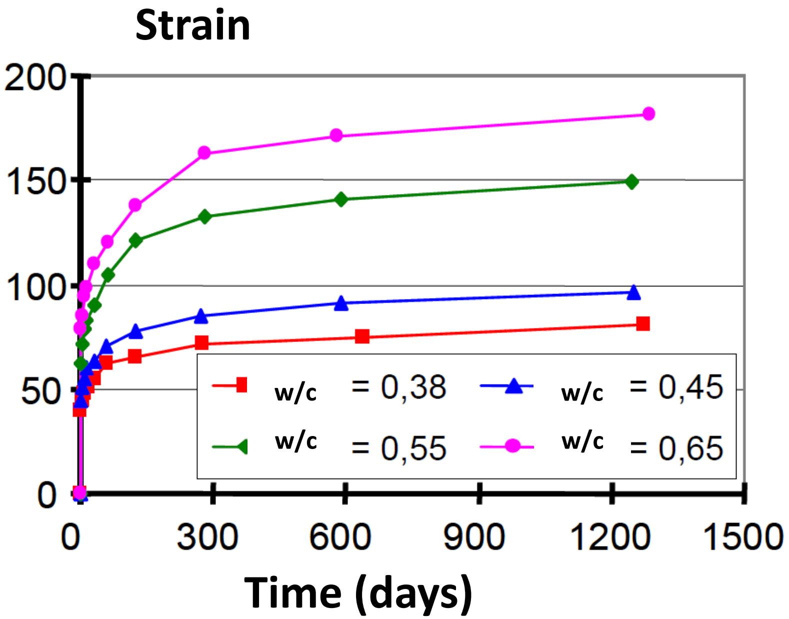
1. Time-dependent behavior: observations in laboratory and in-situ
2. Consideration of time-dependent behavior in design code
3. Time-dependent behavior: linear viscoelastic models
   - Notions of creep compliance and relaxation modulus functions
   - Correspondence principle
   - Rheological models
   - Application of correspondence principle: examples
   - Complex modulus
4. Non-linear viscoelastic and elasto-viscoplastic models

Time-dependent behavior : observations in laboratory and in-situ

1. *Creep test*

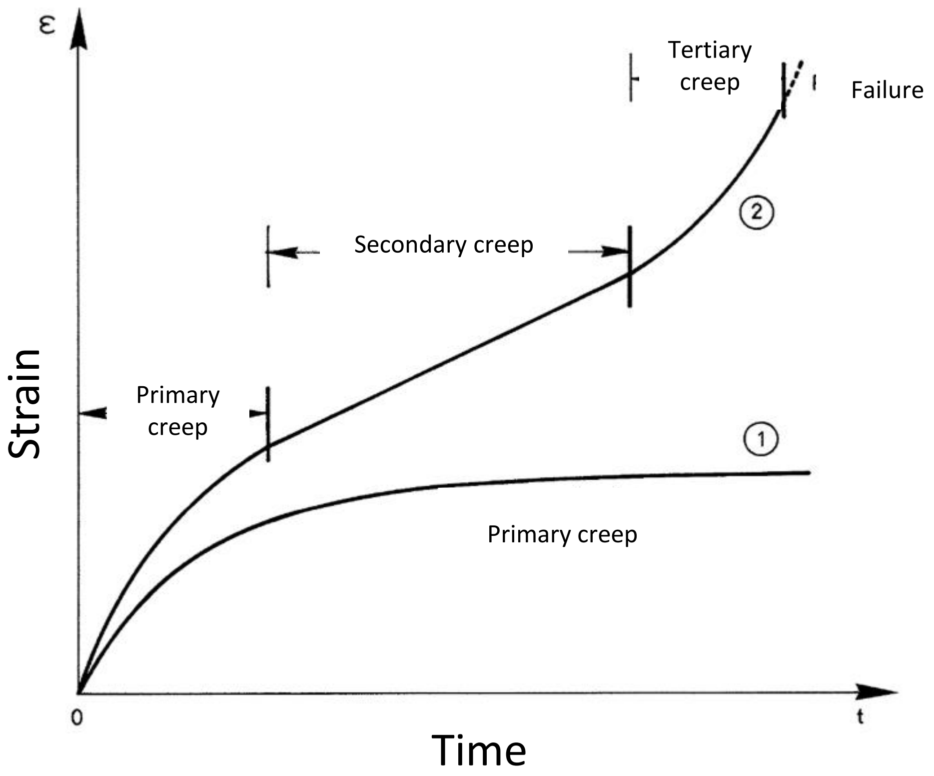
- The principle of this test is to impose a constant stress (uniaxial or triaxial) on the sample of material (in isothermal condition) and the displacements (vertical, horizontal) over time are observed.
    - Increase of displacement (strain) as function of time





Time-dependent behavior : observations in laboratory and in-situ

1. Creep test



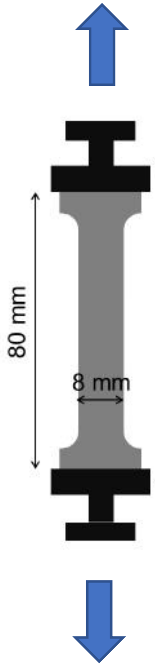
- As function of applied stress: one, two or three creep stages can be observed
  - Primary creep: the creep rate decreases
  - Secondary creep: constant creep rate
  - Tertiary creep: increase of creep rate and approaches failure

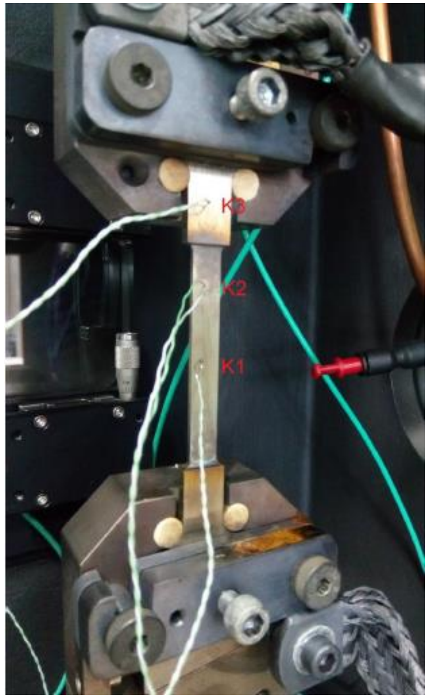
Time-dependent behavior : observations in laboratory and in-situ

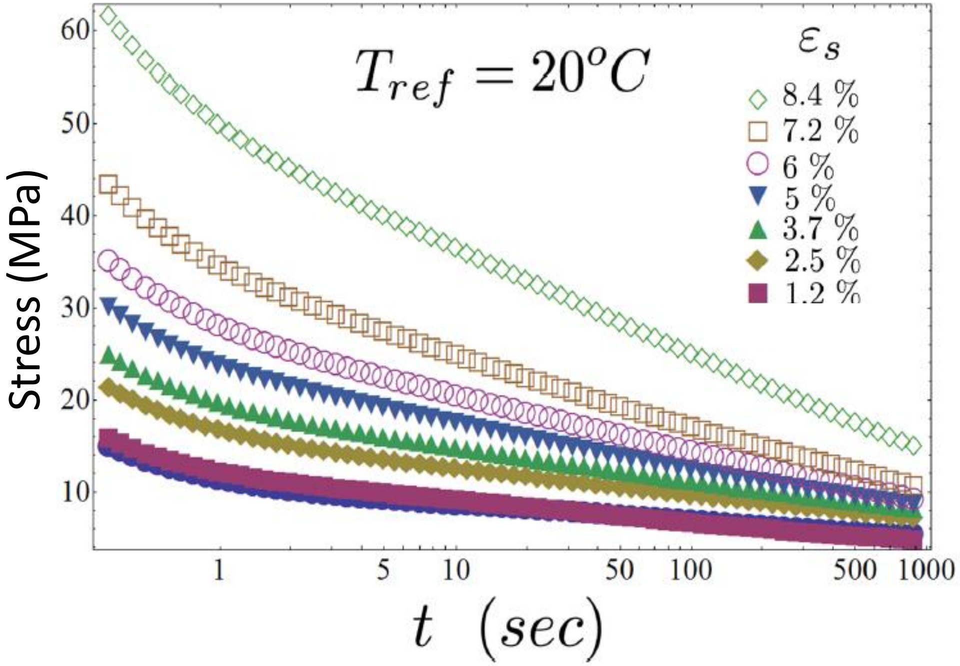
2. Relaxation test

- The principle of the relaxation test is the counterpart of the creep test when one maintains a constant strain (ϵ^0) and observes the evolution as a function of time of the stress .

  - Decrease of stress over time







Time-dependent behavior : observations in laboratory and in-situ

3. In-situ observation

- Important vertical deflection of Savines bridge in France (20cm instead of 3cm)
  - Important vertical deflection after the first year in service (~16cm)
  - Since it was put into service, the bridge has undergone numerous maintenances which frequently consisted of the addition of asphalt at mid-span in order to improve the profile of the roadway.
  - Each of these maintenances then increases the self-weight and consequently the creep deformations.
  - It was decided in 2003 to accept a non-planar condition of the roadway

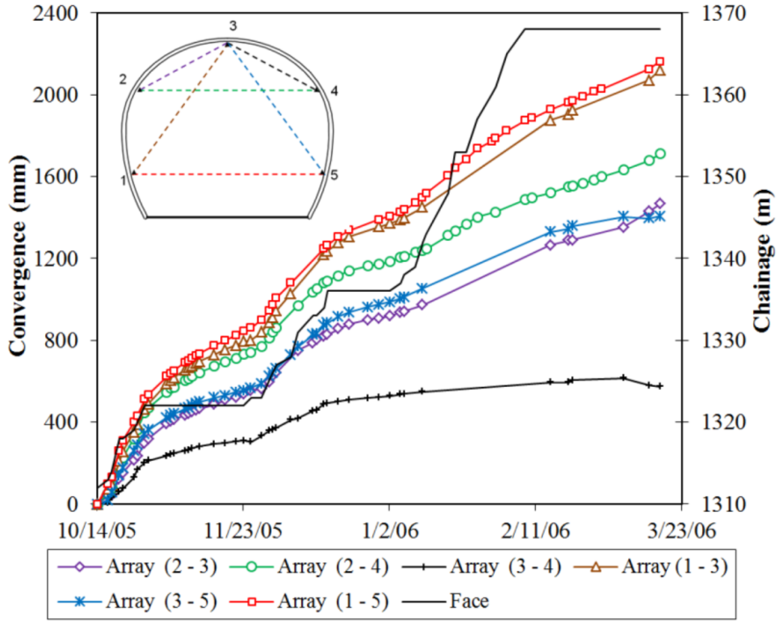
Sellin et al. (2016)

Time-dependent behavior : observations in laboratory and in-situ

3. In-situ observation

- ❑ Construction of underground structure in rock mass
  - ○ Rail tunnel (Lyon-Turin): a strong convergence on the surface of tunnel whose magnitude exceeds 2m after 145 days $\Longrightarrow$ strong time-dependent behavior of rock mass
  - ○ Strong effect on the tunnel design

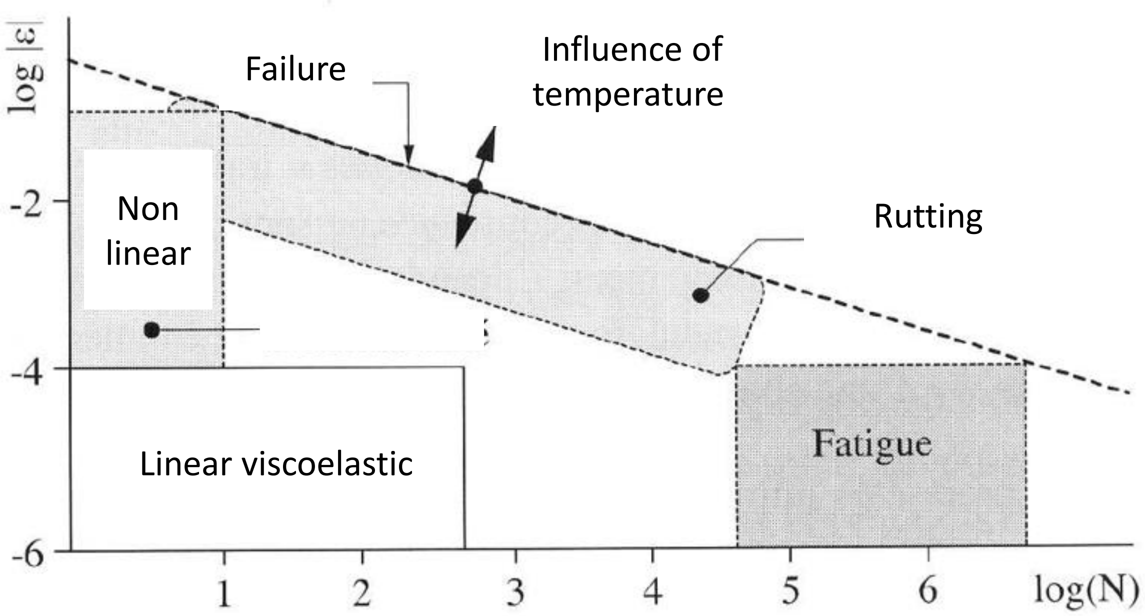
(M.H. Tran 2014)



Time-dependent behavior : observations in laboratory and in-situ

3. In-situ observation

- Rutting bituminous pavements
  - Combination of the time-dependent behavior of bituminous materials under high-cyclic loads and temperature



Consideration of time-dependent behavior in design code

- The time-dependent behavior is taken into account in the design code such as in the reinforced and prestressed concrete structure design. In such structure, the time dependent behavior is a complex combination of the creep and shrinkage effect in concrete as well as relaxation of steel

- The loss of prestress due to relaxation of steel is a well-known time dependent phenomenon in the prestressed concrete. This loss is defined as the percentage ratio of the variation of the prestressing stress over the initial prestressing stress. For instance, according to Eurocode 2 (EC2), this loss is classified as function of class of steel :

$$\frac{\Delta\sigma_{pr}}{\sigma_{pi}} = \begin{cases} 5.39\rho_{1000}\exp(6.7\mu)\left(\frac{t}{1000}\right)^{0.75(1-\mu)} 10^{-5} & \text{class} \quad 1 \\ 0.66\rho_{1000}\exp(9.1\mu)\left(\frac{t}{1000}\right)^{0.75(1-\mu)} 10^{-5} & \text{class} \quad 2 \\ 1.98\rho_{1000}\exp(8\mu)\left(\frac{t}{1000}\right)^{0.75(1-\mu)} 10^{-5} & \text{class} \quad 3 \end{cases}$$

- Coefficient ρ_{1000}: value of relaxation loss (in %) at 1000 hours after tensioning and at a mean temperature of 20°C. According EC2 the long term values of the relaxation losses may be estimated for a time equal to t=500 000 hours (i.e. around 57 years).

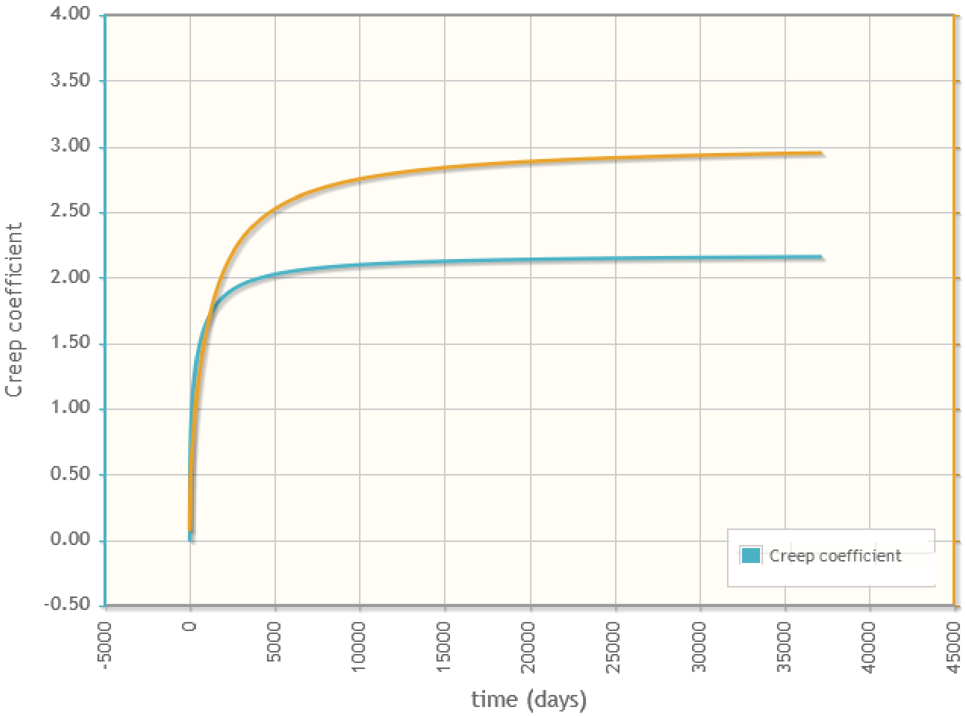
Consideration of time-dependent behavior in design code

- According to EC2, the creep behavior of concrete is characterized by the creep function written in terms of the elastic modulus and creep coefficient

$$\varepsilon_c(t,t_0) = J(t,t_0).\sigma_c, \quad J(t,t_0) = \frac{1}{E(t_0)} + \frac{\varphi(t,t_0)}{E_{28}},$$

$$E(t_0) = E_{28}\sqrt{\exp\left(s\left[1-\sqrt{28/t_0}\right]\right)},$$

$$\varphi(t,t_0) = \frac{\phi_{RH}\beta_f}{0.1+t^{0.2}}\left(\frac{t-t_0}{\beta_H + t - t_0}\right)^{0.3}$$



- To take into account the non-linearity of creep under high compressive stress, a factor is introduced to multiply the creep coeffcient :

$$k_\varphi = \exp\left(1.5(k_\sigma - 0.45)\right), \quad k_\sigma = \frac{\sigma_c}{f_c}$$

Consideration of time-dependent behavior in design code

- This model of creep behavior of concrete is based on average behavior of standard concretes. Even for these concretes, the uncertainty range on the precision is estimated at +/- 30%

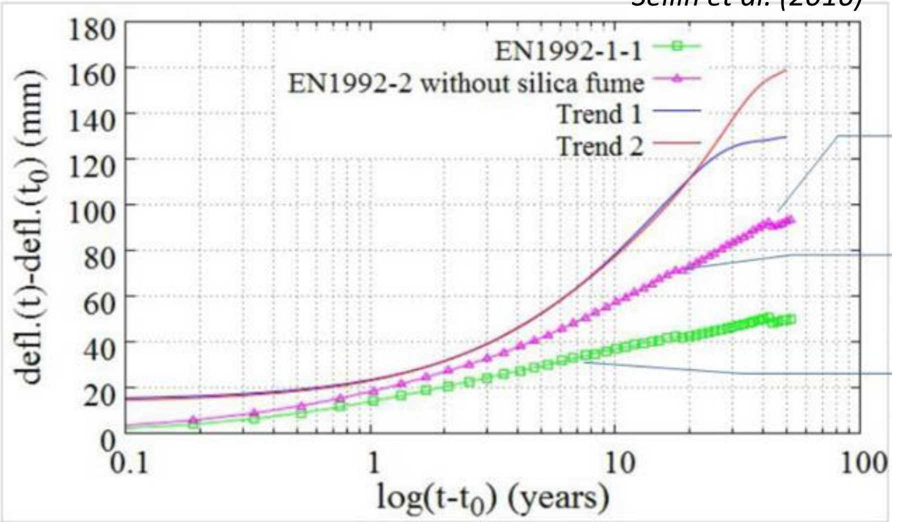
- For concretes with a large quantity of paste, the variations can be from single to double or even beyond

- The formulas and experimental determinations of creep are based on data collected over limited time periods. The extrapolation of such results for very long-term evaluations (~100 years) leads to the additional errors associated with the mathematical expressions used for the extrapolation.

- Important vertical deflection of Savines bridge in France : comparison of in-situ measurment and numerical modeling

  - the deflection obtained with the Eurocodes systematically underestimate the in-situ measurements by around 2 to 3 times

Sellin et al. (2016)



Time-dependent behavior: linear viscoelastic models

Solid or Liquid?

- Long deformation time: pitch behaves like a highly viscous liquid
  - 9th drop fell July 2013
- Short deformation time: pitch behaves like a solid

Started in 1927 by Thomas Parnell in Queensland, Australia

http://www.theatlantic.com/technology/archive/2013/07/the-3-most-exciting-words-in-science-right-now-the-pitch-dropped/277919/

TA | TAINSTRUMENTS.COM

(from Tainstrument.com)

Range of Material Behavior

Liquid Like---------- Solid Like

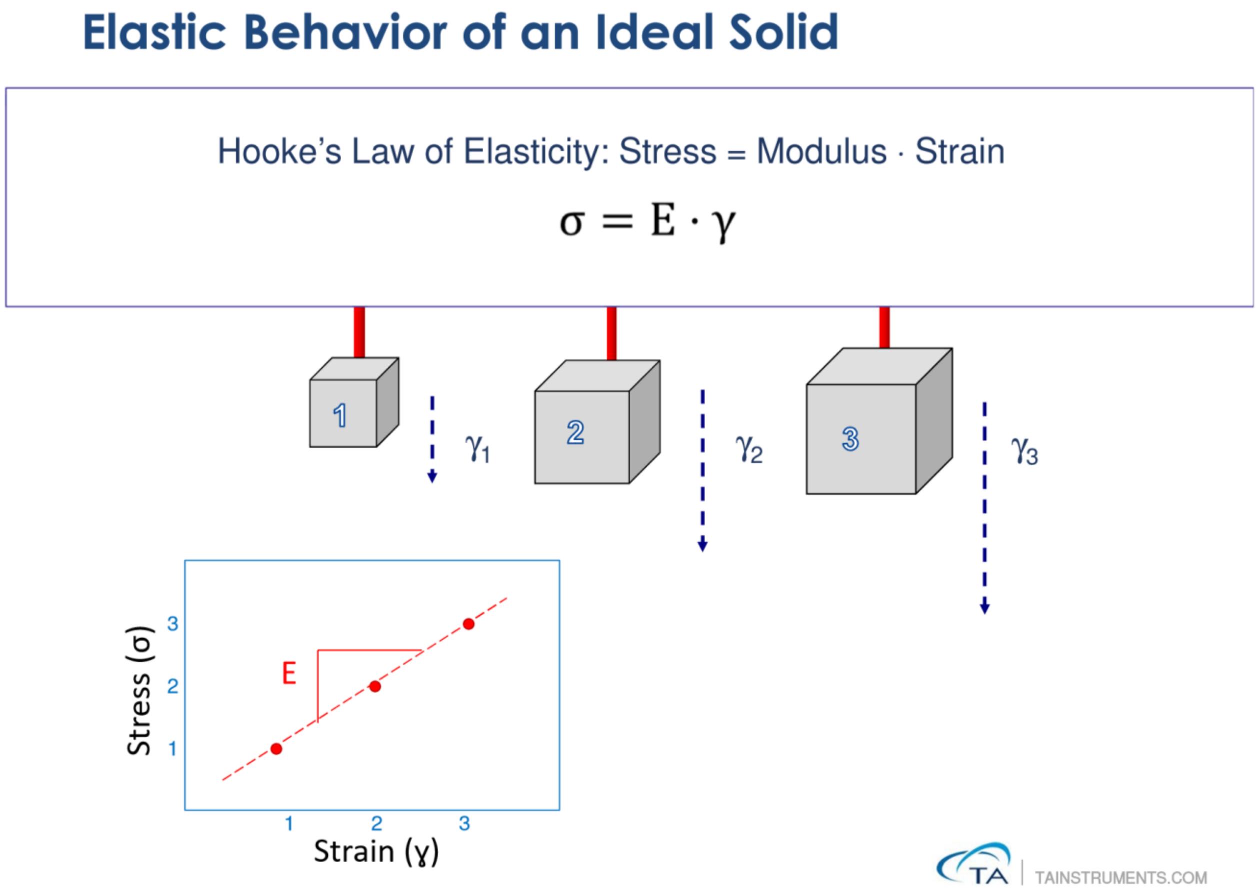
Ideal Fluid ----- Most Materials -----Ideal Solid

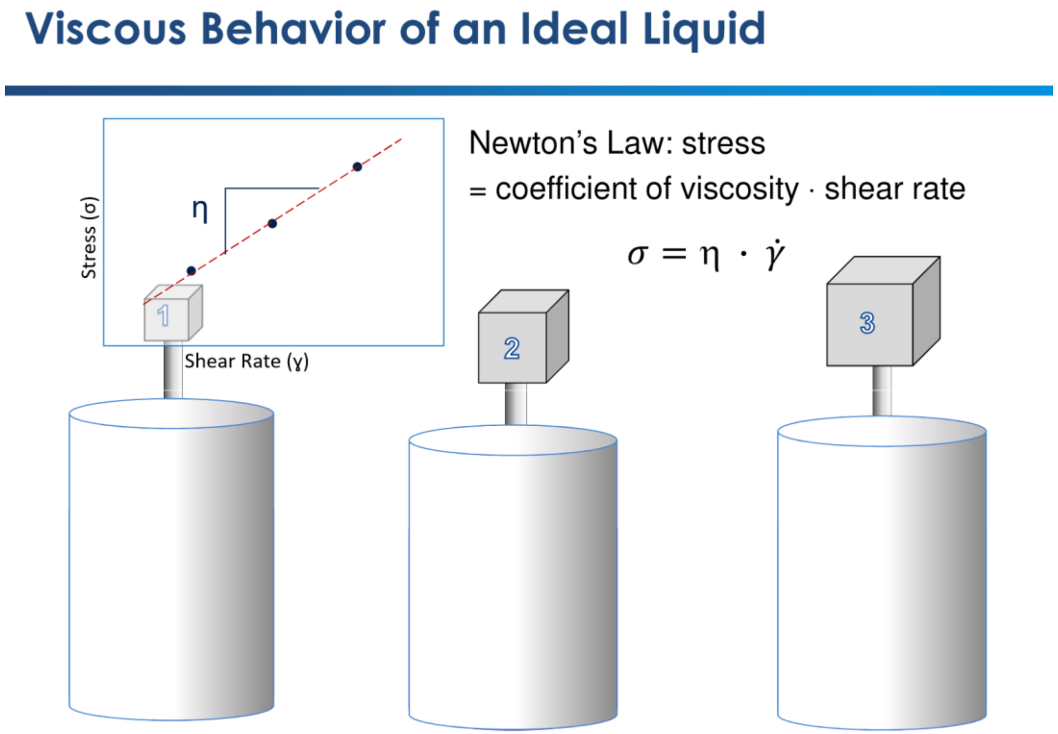
Purely Viscous ----- *Visco*elastic ----- Purely Elastic

Viscoelasticity: Having both viscous and elastic properties

- Materials behave in the linear manner, as described by Hooke and Newton, only on a small scale in stress or deformation.

Time-dependent behavior: linear viscoelastic models





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Time-dependent behavior: linear viscoelastic models

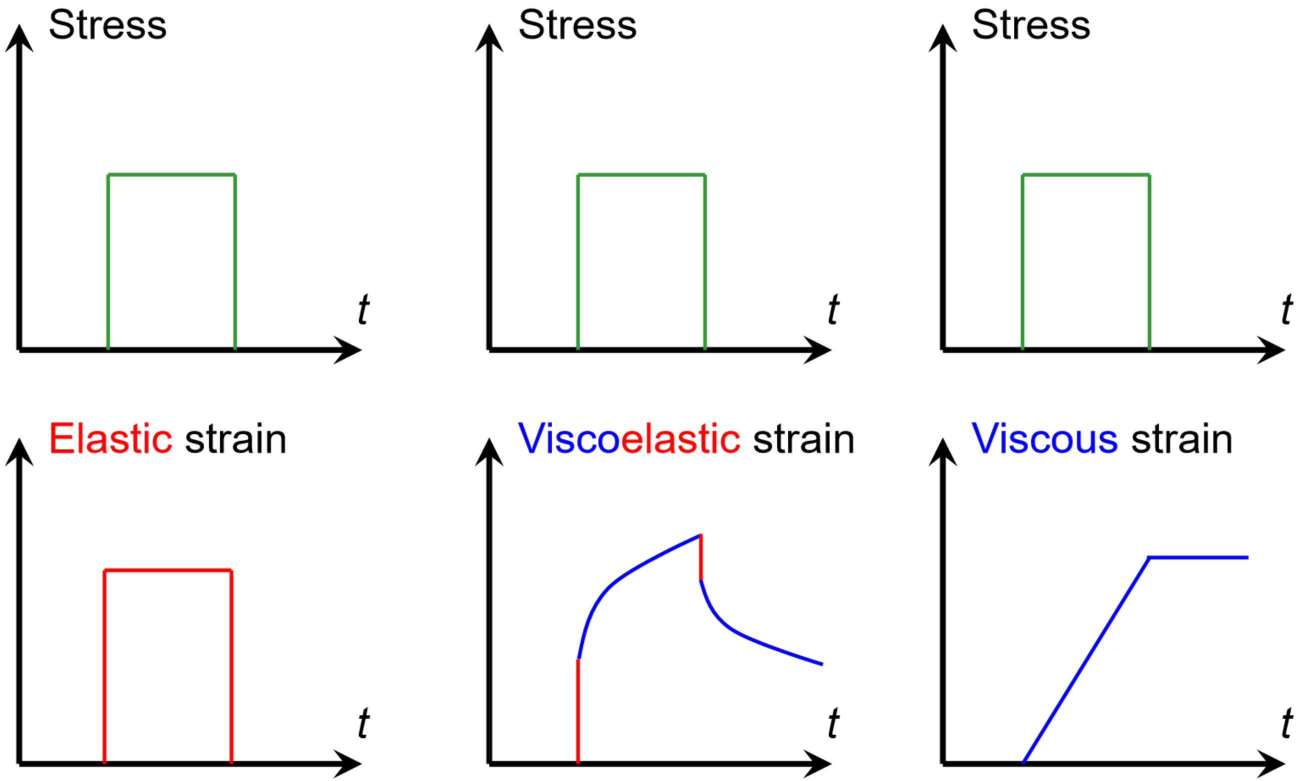
Range of Material Behavior

Liquid Like---------- Solid Like

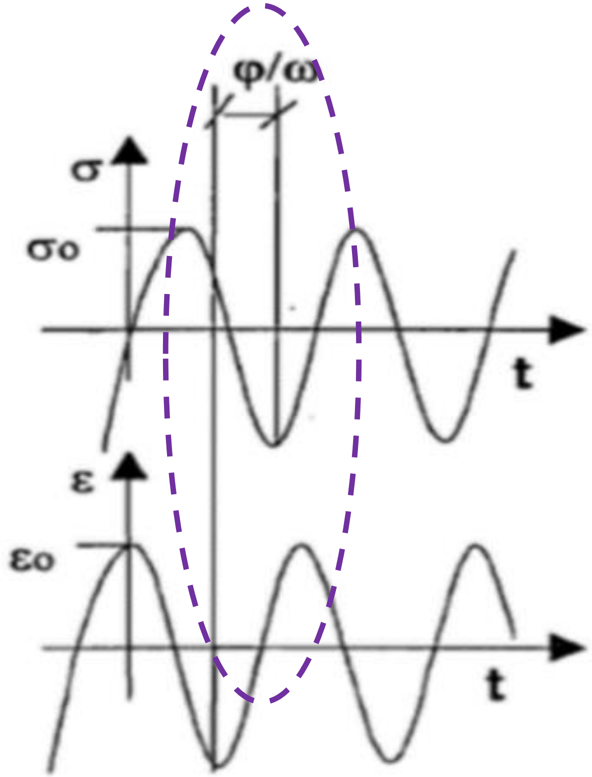
Ideal Fluid ----- Most Materials -----Ideal Solid

Purely Viscous ----- *Visco*elastic ----- Purely Elastic

Viscoelasticity: Having both viscous and elastic properties



Time-dependent behavior: linear viscoelastic models



Phase shift between stress and strain under **harmonic loading**-> viscous behavior of the material ??

Time-dependent behavior: linear viscoelastic models

Sinusoidal loading:

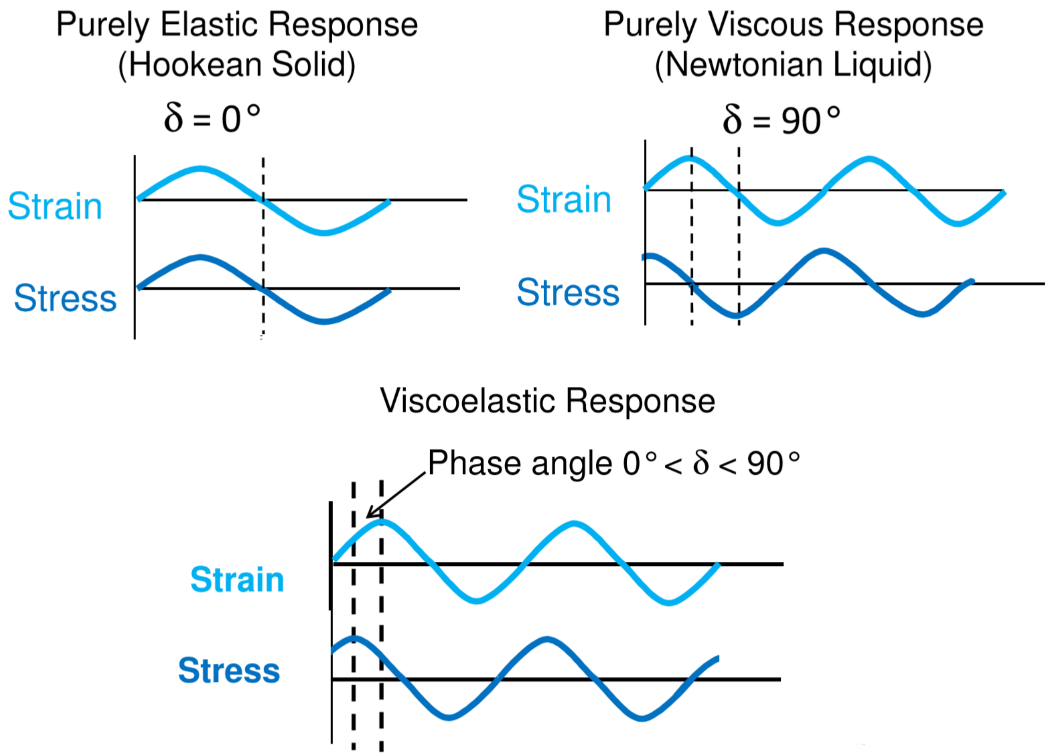
$$\varepsilon(t) = \varepsilon^0 \sin \omega t$$

+ Purely elastic behavior (phase shift: δ = 0)

$$\sigma(t) = E\varepsilon(t) = E\varepsilon^0 \sin \omega t = \sigma^0 \sin \omega t$$

+ Purely viscous behavior (phase shift: δ = 90°)

$$\sigma(t) = \eta.\varepsilon'(t) = \eta\varepsilon^0 \cos \omega t = \eta\varepsilon^0 \sin(\omega t - \tau / 2)$$



+ Phase shift δ characterizes the visco-elastic behavior. When δ approaches $\pi/2$: the closer the behavior of the material is to that of a viscous fluid

Time-dependent behavior: linear viscoelastic models

- Creep compliance function:

$$\sigma(t) = \sigma^0 Y_{t_0}(t)$$

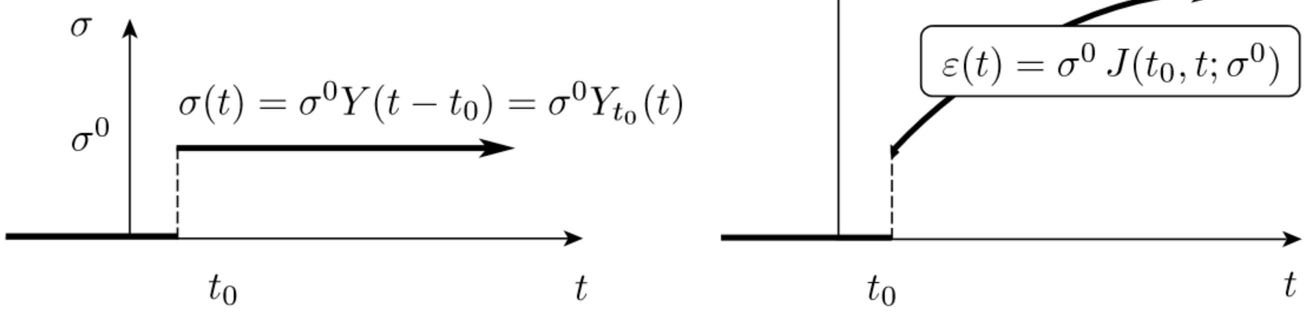
Heaviside function: $$Y_{t_0}(t) = \begin{cases} 0 & t < t_0 \\ 1 & t \geq t_0 \end{cases}$$

$$\varepsilon(t) = \sigma^0 J\left(t_0, t, \sigma^0\right)$$

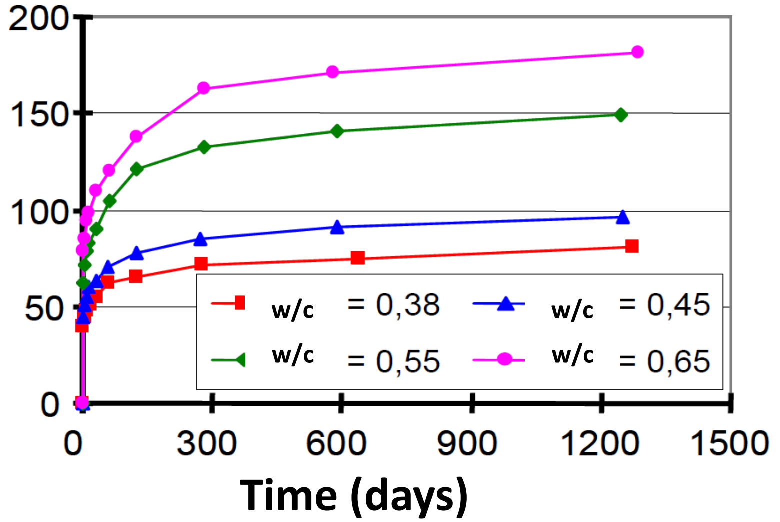
Creep (compliance) function $J(t_0,t)$:

$$\begin{cases} J\left(t_0, t, \sigma^0\right) = 0 & \text{if} \quad t < t_0 \\ J\left(t_0, t_0, \sigma^0\right) > 0 & \text{for} \quad t = t_0 \\ J\left(t_0, t, \sigma^0\right) \ \text{increasing} & \text{for} \quad t \geq t_0 \end{cases}$$

Creep test



Strain



Time-dependent behavior: linear viscoelastic models

- Relaxation modulus function:

$$\varepsilon(t) = \varepsilon^0 Y_{t_0}(t)$$

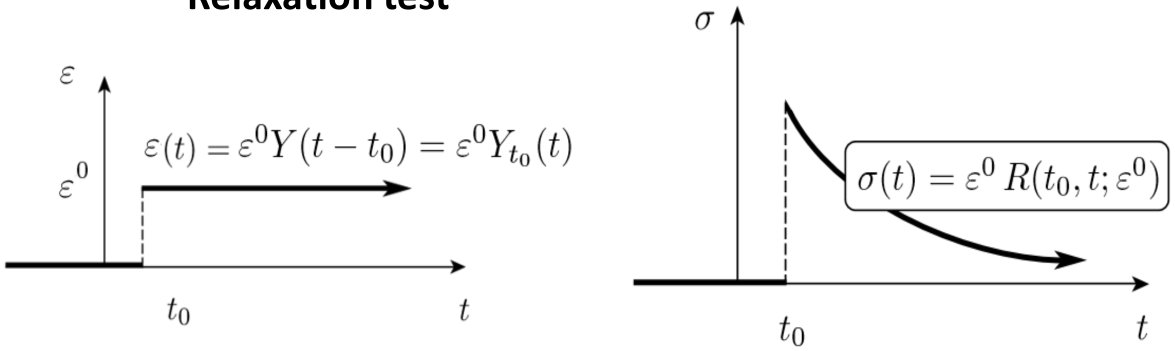
Heaviside function: $$Y_{t_0}(t) = \begin{cases} 0 & t < t_0 \\ 1 & t \geq t_0 \end{cases}$$

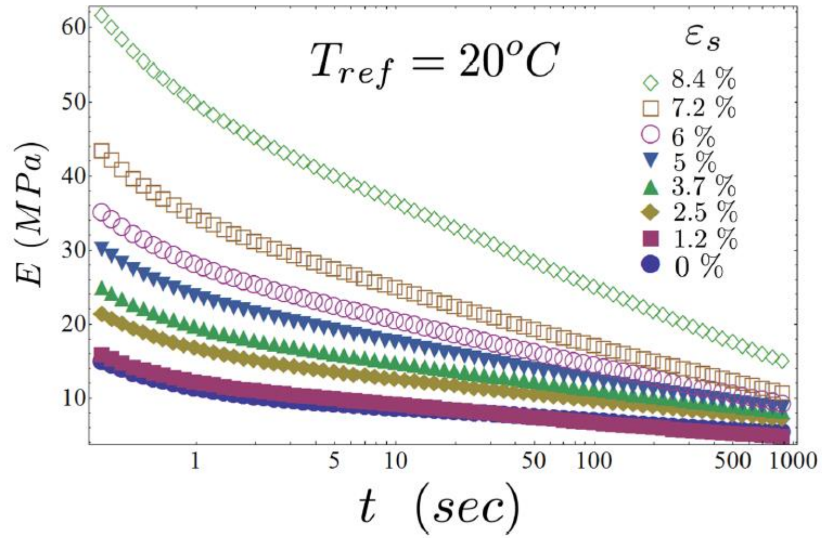
$$\sigma(t) = \varepsilon^0 R\left(t_0, t, \sigma^0\right)$$

Relaxation (modulus) function *R(t₀,t)*:

$$\begin{cases} R\left(t_0, t, \sigma^0\right) = 0 & \text{if} \quad t < t_0 \\ R\left(t_0, t_0, \sigma^0\right) = 0 & \text{for} \quad t = t_0 \\ R\left(t_0, t, \sigma^0\right) \geq 0 & \text{for} \quad t > t_0 \\ R\left(t_0, t, \sigma^0\right) \ \text{decreasing} & \text{for} \quad t > t_0 \end{cases}$$

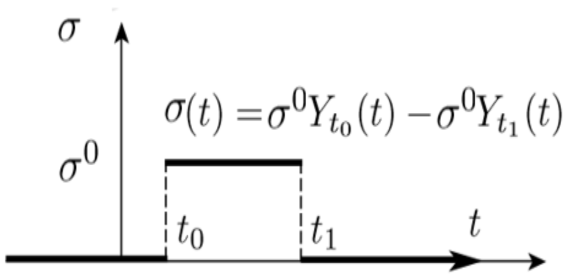
Relaxation test

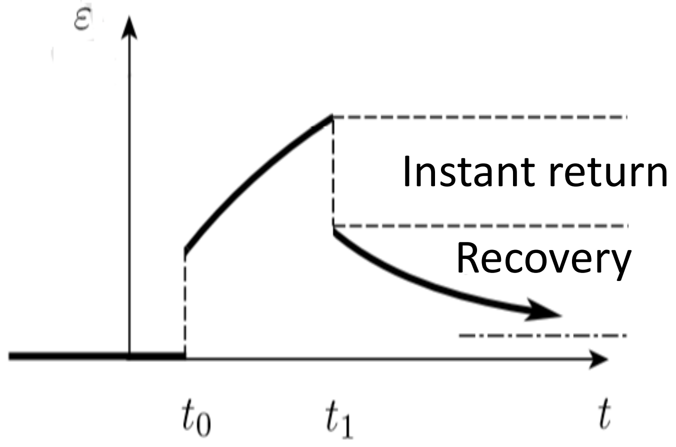




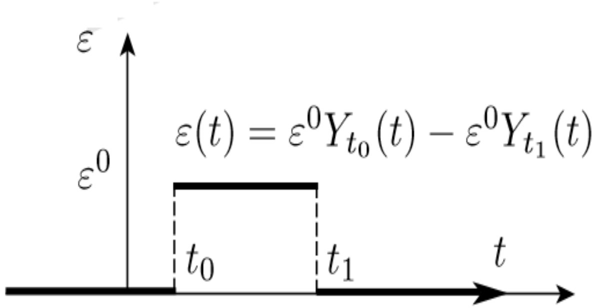
Time-dependent behavior: linear viscoelastic models

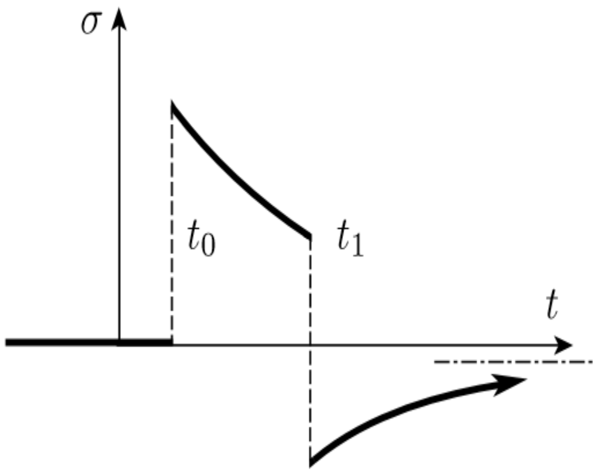
- ❑ Creep recovery test:





- ❑ Relaxation test with unloading:

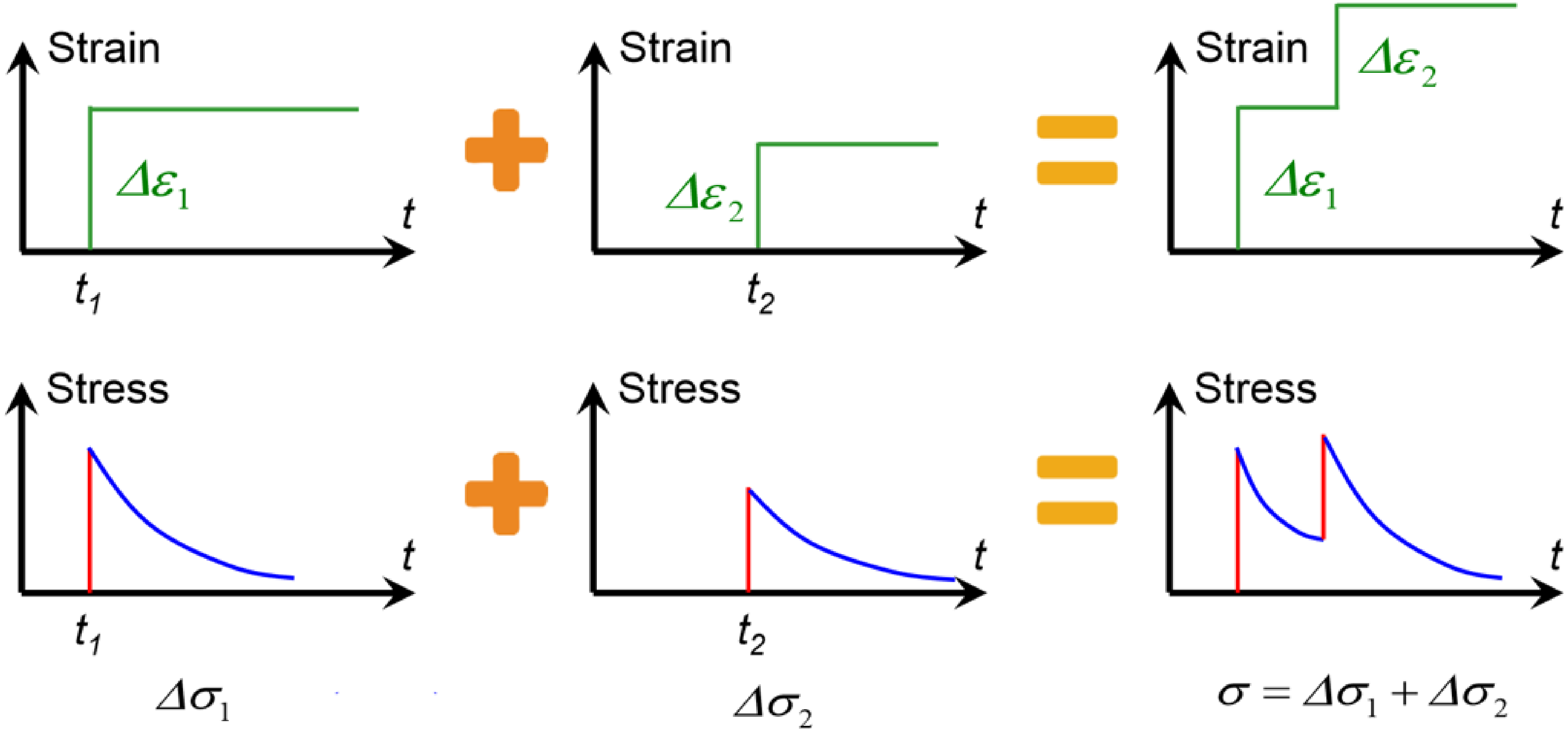




Time-dependent behavior: linear viscoelastic models

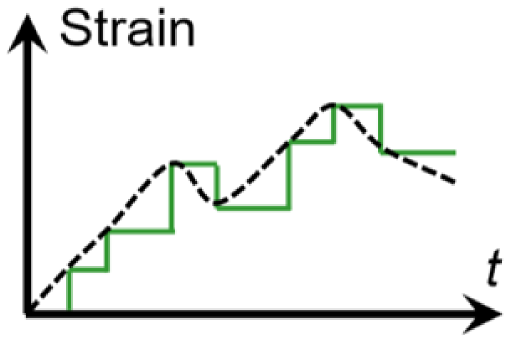
- Boltzmann superposition principle:

  - In the visco-elastic regime: the stress (strain) responses to successive strain (stress) stimuli are additive



Time-dependent behavior: linear viscoelastic models

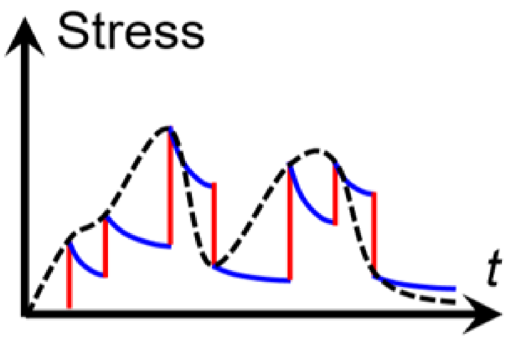
- Stieltjes integral form:



$$\varepsilon = \sum_i \Delta\varepsilon_i$$

$$\varepsilon = \int_0^t \frac{d\varepsilon_i}{dt'} \cdot dt'$$

- Form of the Stieltjes integral with respect to the derivative of the stress



$$\sigma = \sum_i \Delta\sigma_i = \sum_i \Delta\varepsilon_i \cdot r\,(t - t_i)$$

$$\sigma = \int_0^t \frac{d\sigma_i}{dt} \cdot dt = \int_0^t \frac{d\varepsilon}{dt'} \cdot r\,(t - t') \cdot dt'$$

Time-dependent behavior: linear viscoelastic models

- For the non-aging material (i.e. its mechanical properties do not change over time), the creep compliance and relaxation modulus functions are expressed as:

$$J(t_0,t) = f(t-t_0) \quad \text{or} \quad R(t_0,t) = r(t-t_0)$$

- The two functions *f* and *r* verify : $f(0)r(0)=1; \quad f(\infty)r(\infty)=1;$
- The superposition gives the solution:

$$\varepsilon(t) = \int_{-\infty}^{t} J(\tau,t)\sigma'(\tau)d\tau = \sigma(t)J(t,t) - \int_{-\infty}^{t} \sigma(\tau)\frac{\partial J(\tau,t)}{\partial \tau}d\tau$$

$$\varepsilon(t) = \sigma(t)f(0) + \int_{-\infty}^{t} \sigma(\tau)f'(t-\tau)d\tau = \int_{-\infty}^{\infty} \sigma(\tau)f'(t-\tau)d\tau$$

(with: $f(t-\tau)=0$ if $\tau>t$)

- The Riemann convolution product: $\varepsilon = f'*\sigma$ or $\sigma = r'*\varepsilon$

Time-dependent behavior: linear viscoelastic models

- Laplace transformation (*s*: variable in the Laplace's space):

$$L\{g\}(s) = \int_{-\infty}^{\infty} g(t)e^{-st}dt$$

$g(t)$	$L\{g\}(s)$
α	α / s
$H(t)$	$1/s$
$\delta(t-\tau)$	$e^{-\tau s}$
$\dot{\delta}(t)$	s
$e^{-\alpha t}$	$1/(\alpha+s)$
$\left(1-e^{-\alpha t}\right)/\alpha$	$1/s(\alpha+s)$
$t/\alpha-\left(1-e^{-\alpha t}\right)/\alpha^2$	$1/s^2(\alpha+s)$
t^n	$n!/s^{1+n}, \quad n=0,1,\cdots$

Time-dependent behavior: linear viscoelastic models

- The Laplace-Carson transform of the distribution g, noted g_{LC}, is the Laplace transform of the derivative of g:

$$g_{LC}(s) = L\{g'\}(s) = s.L\{g\}(s) = s\int_{-\infty}^{\infty} g(t)e^{-st}dt$$

- One of the essential properties of the Laplace and Laplace-Carson transformation concerns the change of the convolution product into a product:

$$L\{a*b\} = L\{a\}L\{b\}$$

$$(a*b')_{LC} = a_{LC}b_{LC}$$

Thereby: $\varepsilon = f'*\sigma$ → $\varepsilon_{LC}(s) = f_{LC}(s)\sigma_{LC}(s)$

Time-dependent behavior: linear viscoelastic models

- We have a linear relationship between stress-strain in the Laplace-Carson space of the linear viscoelastic material:

$$\varepsilon_{LC}(s) = f_{LC}(s)\sigma_{LC}(s)$$

or

$$\sigma_{LC}(s) = r_{LC}(s)\varepsilon_{LC}(s)$$

with : $r_{LC}(s) f_{LC}(s) = 1$

- These relations are similar to those of a linear elastic material in Laplace Carson space (one-dimensional case) whose Young modulus is $E = r_{LC}(s)$.

- In the absence of aging, the solution of the problem of evolution over time (the creep or relaxation experiment) of a linear viscoelastic behavior of the material is similar to an equation of linear elastic behavior in Laplace-Carson space

Time-dependent behavior: linear viscoelastic models

❑ Correspondence principle of linear visco-elasticity theory:

- Under the assumption of small disturbances, the static problem with the ***boundary conditions invariable in time*** of a material whose behavior evolves linearly according to time is written in the space of Laplace-Carson is formally identical to a problem linear elastic equilibrium

- If the explicit (analytical) solution of the linear elastic equilibrium problem exists, it is directly applied to deduce the solution of the original problem of evolution in Laplace-Carson space

- An inverse Laplace-Carson transformation will allow us to deduce the solution sought in time space.

The principle of applying the solution of the linear elastic problem to derive the solution of the problem of evolution in Laplace-Carson space is known as the ***correspondence theorem***.

Time-dependent behavior: linear viscoelastic models

❑ Extension of the linear visco-elasticity theory in 3D:

- In the general case in 3D, the stress and strain expressions will be written in form of tensor. For example, the creep and relaxation functions will be reformulated in the form:

$$\underline{\underline{\varepsilon}}(t)=\underline{\underline{\underline{\underline{J}}}}(t_0,t):\underline{\underline{\sigma}}^{0}=\underline{\underline{\underline{\underline{f}}}}(t-t_0):\underline{\underline{\sigma}}^{0}$$

or

$$\underline{\underline{\sigma}}(t)=\underline{\underline{\underline{\underline{R}}}}(t_0,t):\underline{\underline{\varepsilon}}^{0}=\underline{\underline{\underline{\underline{r}}}}(t-t_0):\underline{\underline{\varepsilon}}^{0}$$

The Riemann convolution product:

$$\underline{\underline{\varepsilon}}=\underline{\underline{\underline{\underline{f}}}}'(*:)\underline{\underline{\sigma}}$$

$$\underline{\underline{\sigma}}=\underline{\underline{\underline{\underline{r}}}}'(*:)\underline{\underline{\varepsilon}}$$

Laplace Carson transform:

$$\underline{\underline{\varepsilon}}_{LC}(s)=\underline{\underline{\underline{\underline{f}}}}_{LC}(s):\underline{\underline{\sigma}}_{LC}(s)$$

$$\underline{\underline{\sigma}}_{LC}(s)=\underline{\underline{\underline{\underline{r}}}}_{LC}(s):\underline{\underline{\varepsilon}}_{LC}(s)$$

$$\underline{\underline{\underline{\underline{f}}}}_{LC}(s):\underline{\underline{\underline{\underline{r}}}}_{LC}(s)=\underline{\underline{\underline{\underline{I}}}}$$

Time-dependent behavior: linear viscoelastic models

❑ Extension of the linear visco-elasticity theory in 3D:

- In the case of the isotropic non-aging linear viscoelastic material, we have Hooke's law in the Laplace-Carson space:

$$\underline{\underline{\varepsilon}}_{LC} = \frac{1+\nu_{LC}}{E_{LC}} : \underline{\underline{\sigma}}_{LC} - \frac{\nu_{LC}}{E_{LC}} \left[\mathrm{tr}\,\underline{\underline{\sigma}}_{LC} \right] \underline{\underline{\delta}}$$

$$\underline{\underline{\sigma}}_{LC} = \lambda_{LC} \left[\mathrm{tr}\,\underline{\underline{\varepsilon}}_{LC} \right] \underline{\underline{\delta}} + 2\mu_{LC}\,\underline{\underline{\varepsilon}}_{LC}$$

- In practice, very often it is considered that the Poisson's ratio of the material is constant:

$$\underline{\underline{\varepsilon}}_{LC} = \frac{1+\nu}{E_{LC}} : \underline{\underline{\sigma}}_{LC} - \frac{\nu}{E_{LC}} \left[\mathrm{tr}\,\underline{\underline{\sigma}}_{LC} \right] \underline{\underline{\delta}}$$

$$\underline{\underline{\sigma}}_{LC} = \lambda_{LC} \left[\mathrm{tr}\,\underline{\underline{\varepsilon}}_{LC} \right] \underline{\underline{\delta}} + 2\mu_{LC}\,\underline{\underline{\varepsilon}}_{LC}$$

with :

$$2\mu_{LC} = \frac{E_{LC}}{1+\nu}, \quad 3\lambda_{LC} + 2\mu_{LC} = \frac{E_{LC}}{1+2\nu}$$

Time-dependent behavior: linear viscoelastic models

Extension of the linear visco-elasticity theory in 3D:

- Correspondence principle of linear visco-elasticity theory:

  - under the boundary conditions S_U and S_T which are independent of time t:

$$S_U \cap S_T = 0, \quad S_U \cup S_T = \partial\Omega$$

  - the equations of the mechanical equilibrium problem in Laplace-Carson space:

$$\mathrm{div}\underline{\underline{\sigma}}_{LC}(s) + \rho \underline{F}_{LC}(s) = 0$$

$$\underline{\underline{\varepsilon}}_{LC}(s) = \frac{1}{2}\left[\underline{\underline{\mathrm{grad}\,\xi_{lC}}}(s) + {}^{t}\,\underline{\underline{\mathrm{grad}\,\xi_{lC}}}(s)\right]$$

  - Solving the problem in Laplace-Carson space as the linear elastic problem

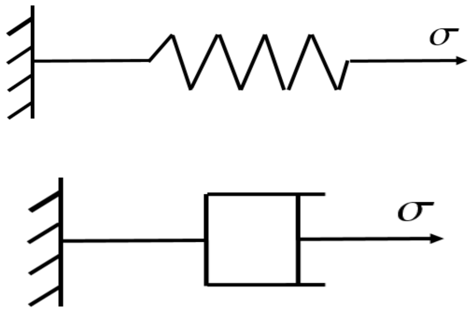
$$\underline{\underline{\varepsilon}}_{LC}(s) = \frac{1+\nu}{E_{LC}} : \underline{\underline{\sigma}}_{LC}(s) - \frac{\nu}{E_{LC}}\left[\mathrm{tr}\,\underline{\underline{\sigma}}_{LC}(s)\right]\underline{\underline{\delta}}$$

➡ Solution in temporal space: inverse Laplace-Carson transformation

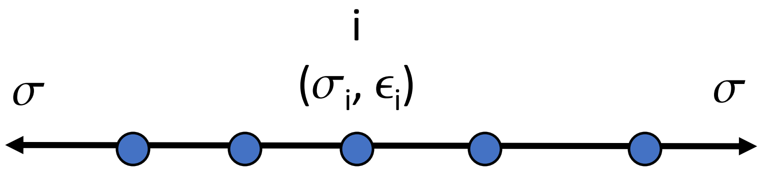
Time-dependent behavior: linear viscoelastic models

❑ Rheological models to present the viscoelastic behavior of the material:

- In general, the rheological models used in linear viscoelasticity for the non-aging material are built from the set of ***two basic elements***: ***a spring to present the linear elastic behavior*** and ***a dashpot (piston-cylinder filled with viscous fluid) to characterize the linear viscous behavior***.

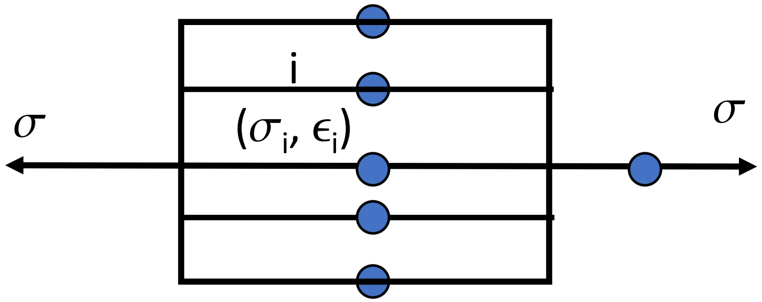


- These basic elements can be constructed:
  - In series: each element undergoes the same stress state and the deformation is the sum of the element strains



$$\forall i \quad \sigma_i = \sigma \qquad \varepsilon = \sum_{i=1}^{n} \varepsilon_i$$

  - In parallel: total stress is the sum of the element stress whilst each element undergoes the same strain

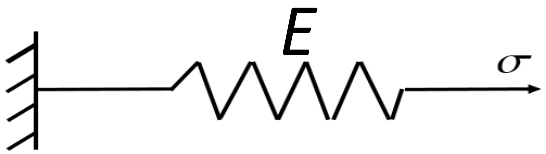


$$\forall i \quad \varepsilon_i = \varepsilon \qquad \sigma = \sum_{i=1}^{n} \sigma_i$$

Time-dependent behavior: linear viscoelastic models

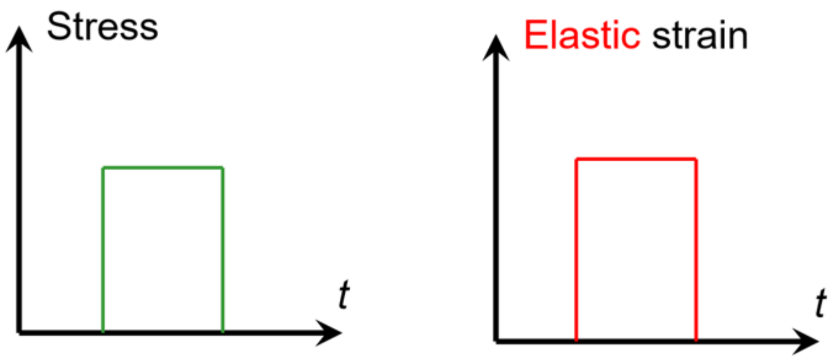
❑Rheological model to present the viscoelastic behavior of the material:

- The model of the spring with constant stiffness E is used to present the linear elastic behavior of the material



$$\sigma(t) = E.\varepsilon(t)$$

- Using this spring model, we obtain the creep and relaxation functions in space of time and Laplace-Carson space as follows:



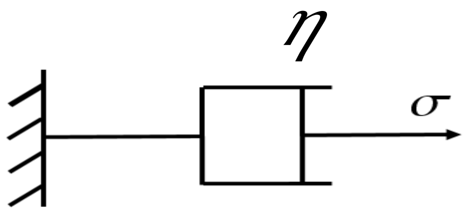
$$\varepsilon = f'*\sigma \qquad \text{or} \qquad \varepsilon_{LC}(s) = f_{LC}(s)\sigma_{LC}(s)$$

Heaviside function:

$$f(t) = \frac{Y(t)}{E}, \quad f_{LC} = \frac{1}{E} \qquad r(t) = E.Y(t), \quad r_{LC} = E \qquad Y_{t_0}(t) = \begin{cases} 0 & t < t_0 \\ 1 & t \geq t_0 \end{cases}$$

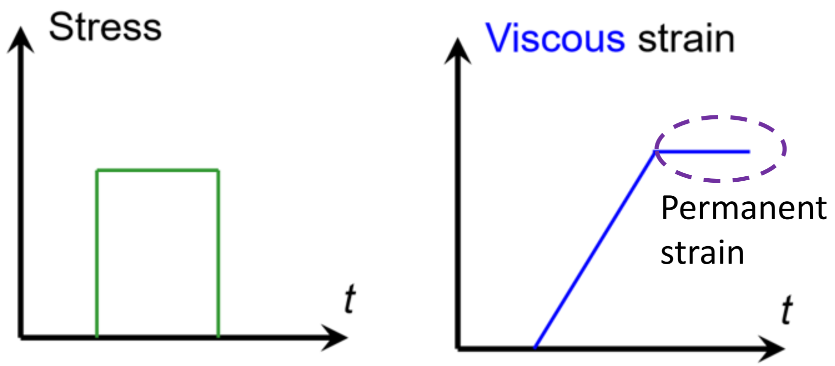
Time-dependent behavior: linear viscoelastic models

- Rheological model to present the viscoelastic behavior of the material:
  - The linear viscous behavior is modeled by the dashpot whose time-independent viscosity coefficient η is defined from the relationship between the stress and the strain rate (or derivative of strain with respect to time):



$$\sigma(t) = \eta.\varepsilon'(t)$$

  - Using this dashpot model, we obtain the creep and relaxation functions in space of time and Laplace-Carson space as follows:



$$\varepsilon = f'^{*}\sigma \qquad \text{or} \qquad \varepsilon_{LC}(s) = f_{LC}(s)\sigma_{LC}(s)$$

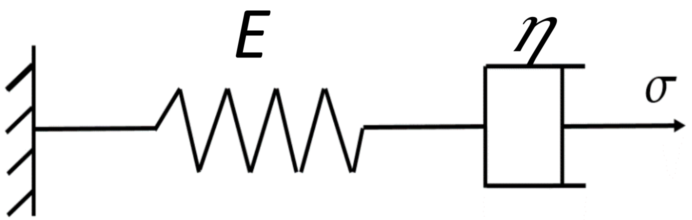
$$f(t) = \frac{t.Y(t)}{E}, \quad f_{LC} = \frac{1}{\eta.s} \qquad r(t) = \eta.s, \quad r_{LC} = \eta.\delta$$

Time-dependent behavior: linear viscoelastic models

- Rheological model to present the viscoelastic behavior of the material:

Maxwell model

- Maxwell's model consists of a spring connected in series with a dashpot:



- Using this model for the creep test

$$\varepsilon'(t) = \varepsilon_e^{'}(t) + \varepsilon_v^{'}(t) = \frac{\sigma'(t)}{E} + \frac{\sigma(t)}{\eta} \quad \Rightarrow \quad \varepsilon_{LC} = \frac{\sigma_{LC}}{E} + \frac{\sigma_{LC}}{s.\eta} = \sigma_{LC}\left(\frac{1}{E} + \frac{1}{s.\eta}\right) = f_{LC}\sigma_{LC}$$

- Creep function:

$$f_{LC} = \frac{1}{E} + \frac{1}{s.\eta} \quad \Rightarrow \quad f(t) = \frac{1}{E} + \frac{t}{\eta}$$

Time-dependent behavior: linear viscoelastic models

❑ Rheological model to present the viscoelastic behavior of the material:

Maxwell model

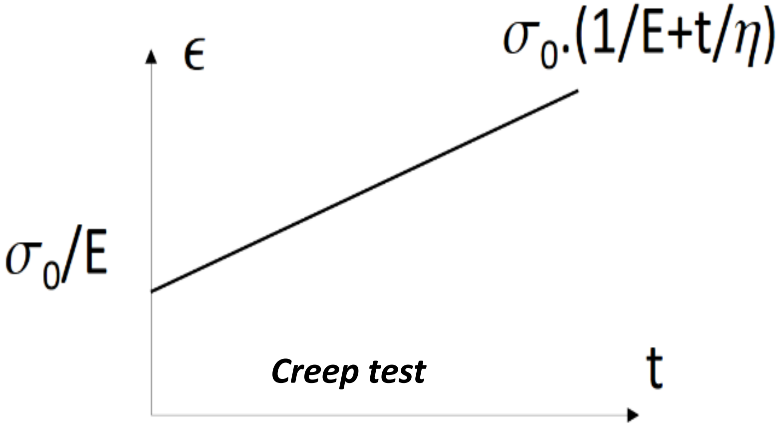
- Relaxation function:

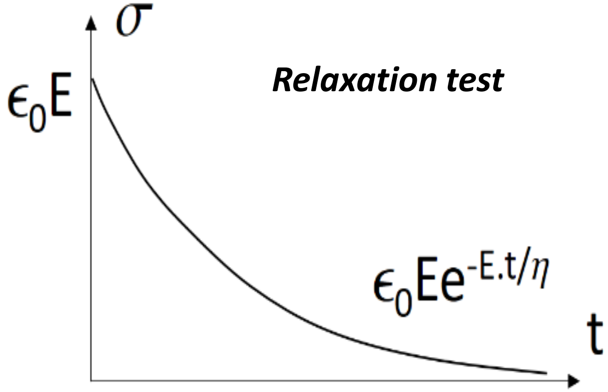
$$r_{LC} = \frac{1}{f_{LC}} = \frac{E.s.\eta}{\eta s + E} \quad \Longrightarrow \quad r(t) = E.e^{-\frac{E.t}{\eta}}$$

- Response to creep and relaxation tests:

  ✓ instataneous (t=0) :
  r(t=0)=E, or *f(0)=1/E*

  ✓ to infinity ($t \longrightarrow \infty$):
  $f(t) \longrightarrow \infty$, $r(t) \longrightarrow 0$



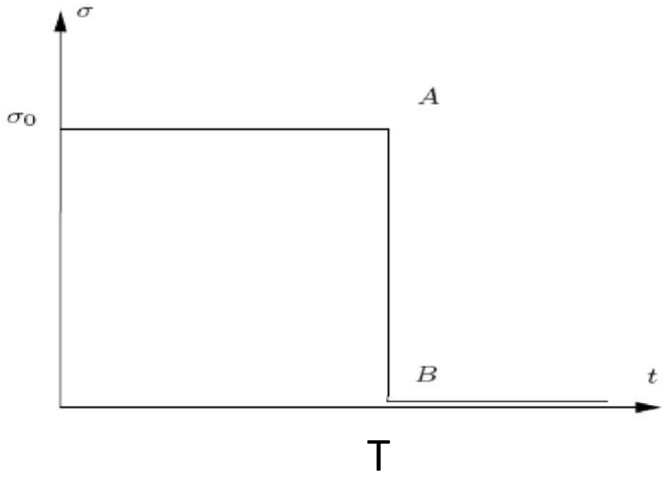


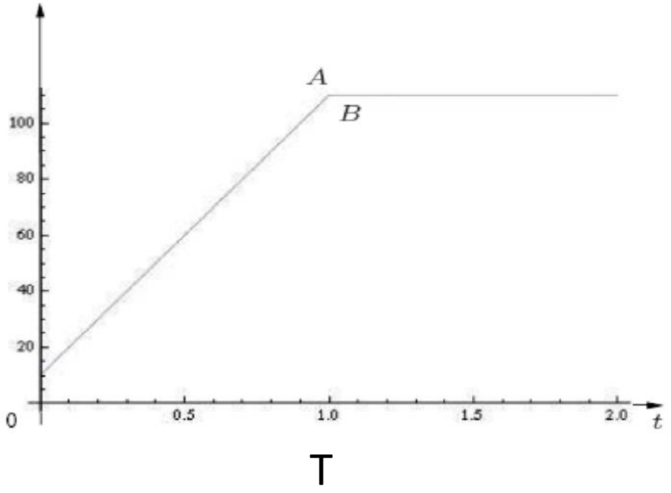
Time-dependent behavior: linear viscoelastic models

Maxwell model

- Response to the creep recovery test:

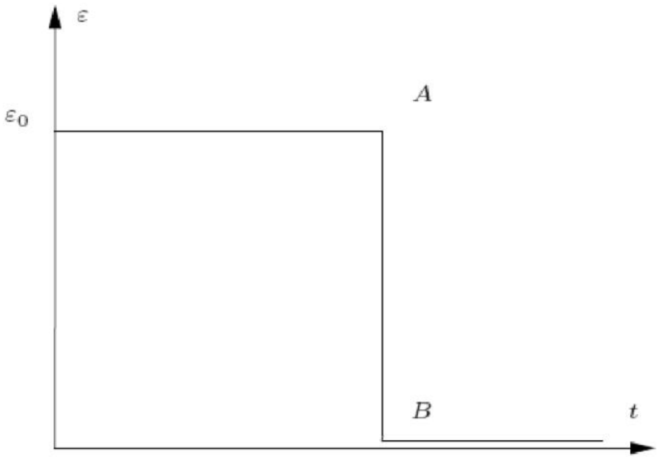
$$\varepsilon(t) = \begin{cases} \dfrac{\sigma_0}{E}\left(1+\dfrac{t}{\tau}\right) & \text{if} \quad 0 \le t \le T \\ \varepsilon(T) & \text{if} \quad t > T \end{cases}$$

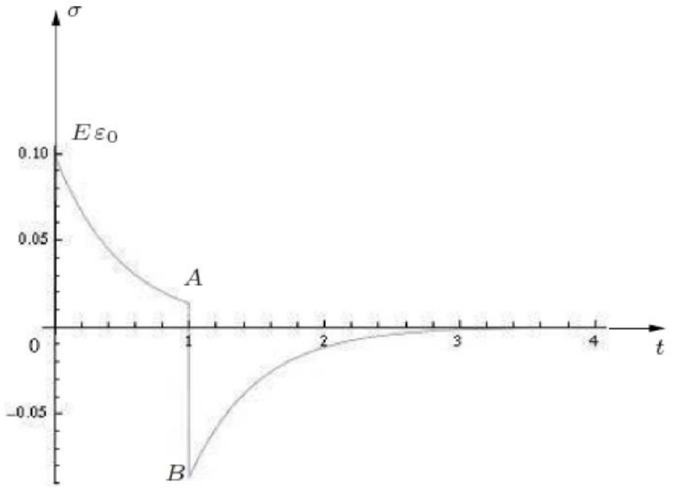




- Response to the relaxation test with loading and unloading:

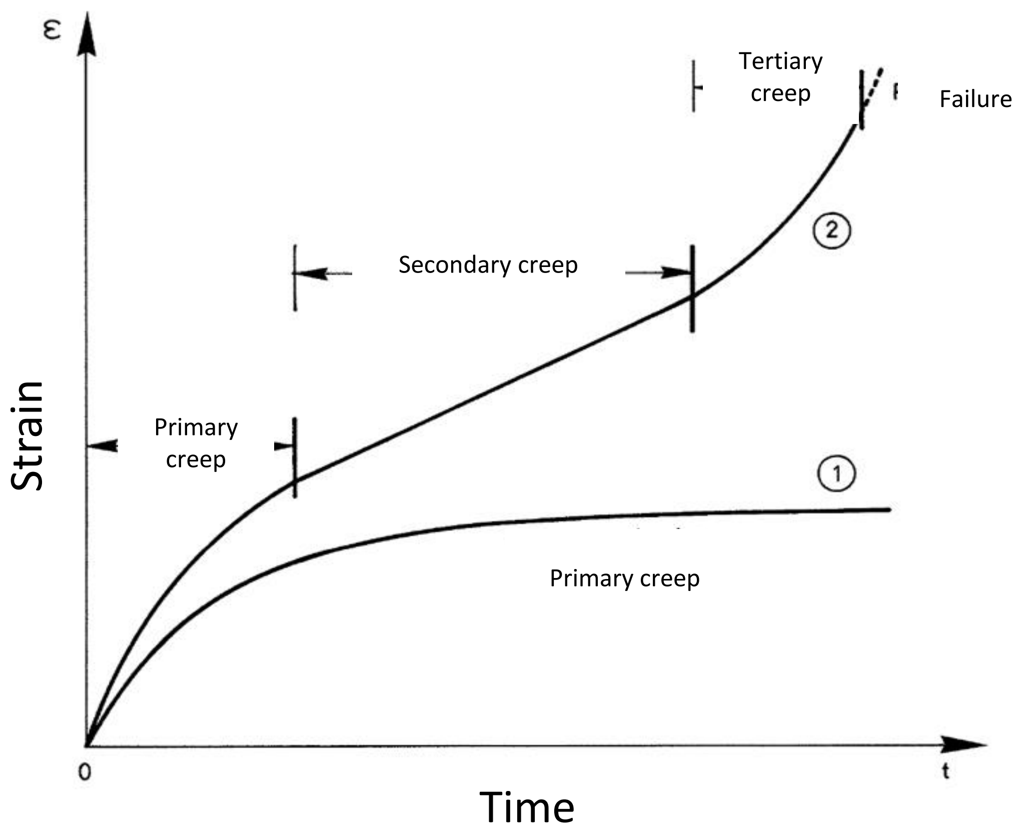
$$\sigma(t) = \begin{cases} E\varepsilon_0 \exp\left(-\dfrac{t}{\tau}\right) & \text{if} \quad 0 \le t \le T \\ E\varepsilon_0 \left[\exp\left(-\dfrac{T}{\tau}\right) - 1\right] \exp\left(-\dfrac{t-T}{\tau}\right) & \text{if} \quad t > T \end{cases}$$



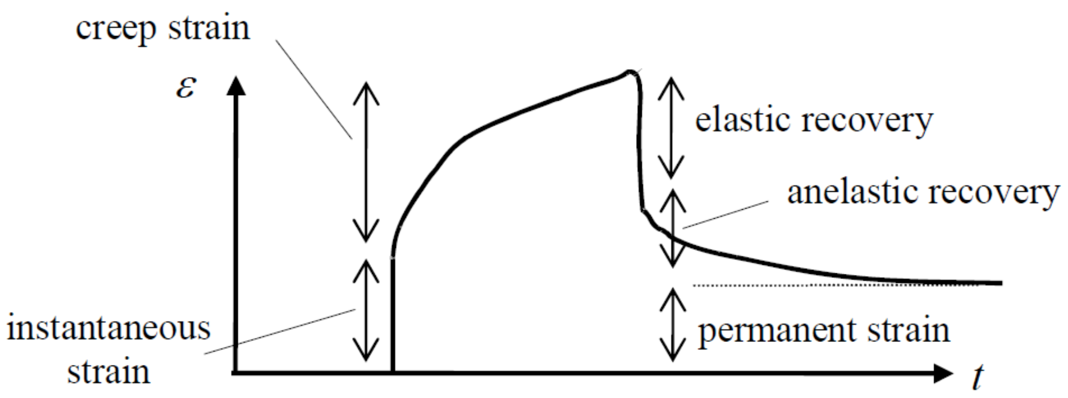


Time-dependent behavior : observations in laboratory and in-situ

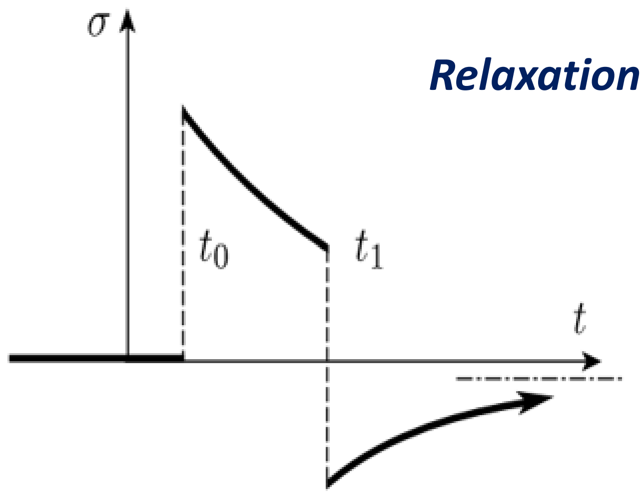
Creep test



creep recovery test

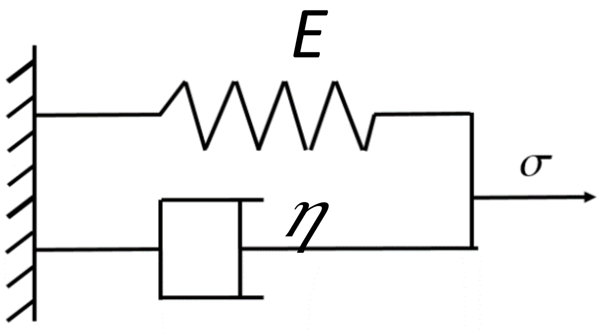


Relaxation test



Time-dependent behavior: linear viscoelastic models

Kelvin model



- The Kelvin model consists of a spring connected in parallel with a dashpot
- Using this model for the relaxation test:

$$\sigma(t)=\sigma_e(t)+\sigma_v(t)=E\varepsilon(t)+\eta.\varepsilon'(t)$$

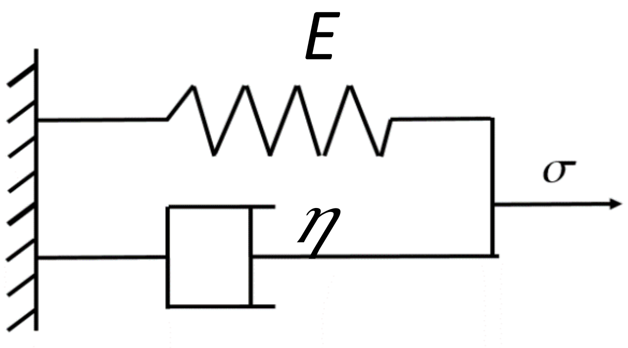
$$\Rightarrow \quad \frac{\sigma_{LC}}{s}=E\frac{\varepsilon_{LC}}{s}+\eta\varepsilon_{LC}=\varepsilon_{LC}\left(\frac{E}{s}+\eta\right)=\frac{r_{LC}.\varepsilon_{LC}}{s}$$

- relaxation function:

$$r_{LC}=E+\eta s \quad \Rightarrow \quad r(t)=E.Y(t)+\eta.\delta(t)$$

Time-dependent behavior: linear viscoelastic models

Kelvin model



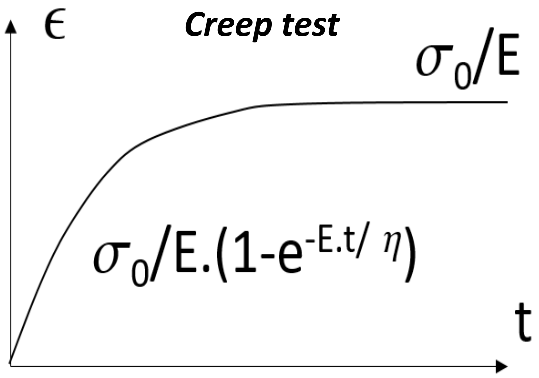
- Creep function:

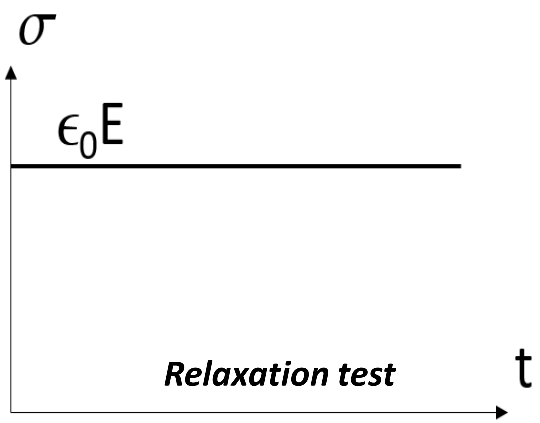
$$f_{LC} = \frac{1}{r_{LC}} = \frac{1}{\eta s + E}, \quad f(t) = \frac{1}{E}(1 - e^{-\frac{E}{\eta}t})$$

- Response to creep and relaxation tests:

✓ instataneous (t=0) :
f(0)=0 ⟶ this type of material does not have instantaneous elasticity

✓ to infinity ($t \longrightarrow \infty$):
$\epsilon(t)=\sigma_0/E$ (creep test) whilst $\sigma(t)=\epsilon_0.E$ (relaxation test).



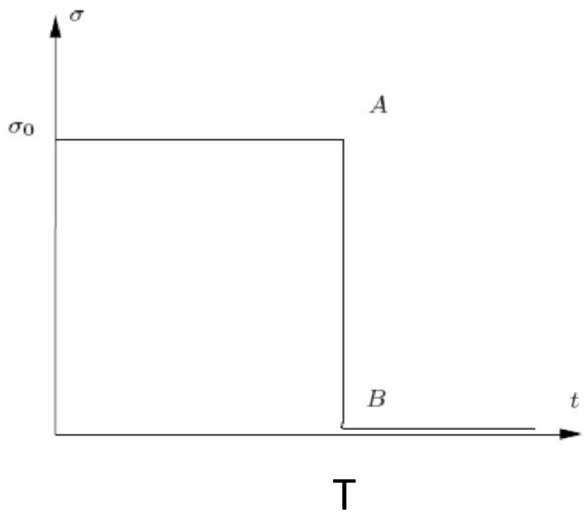


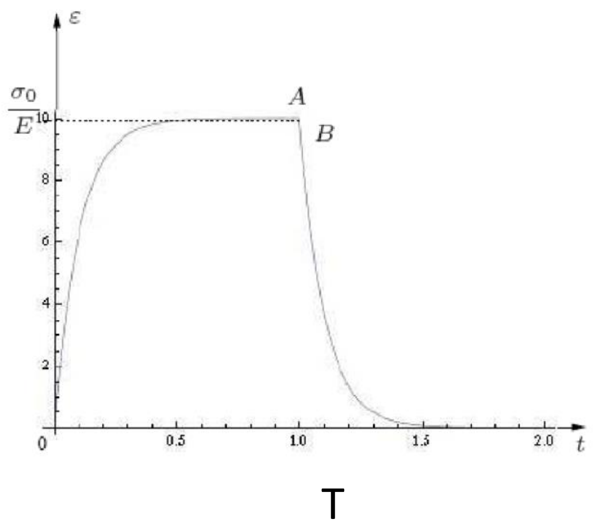
Time-dependent behavior: linear viscoelastic models

Kelvin model

- Response to the creep recovery test:

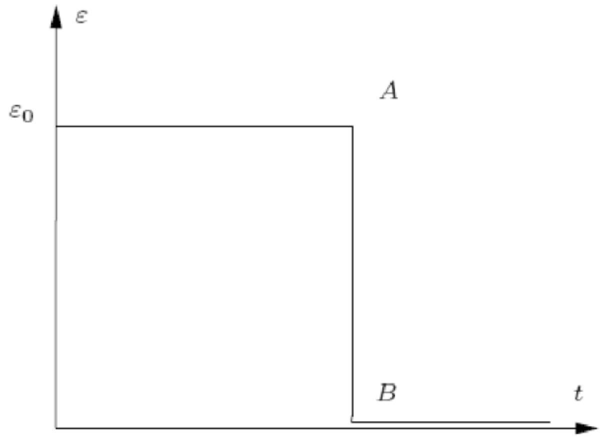
$$\varepsilon(t) = \begin{cases} \dfrac{\sigma_0}{E}\left[1-\exp\left(-\dfrac{t}{\tau}\right)\right] & \text{if } \; 0 \le t \le T \\ \varepsilon(T)\exp\left(-\dfrac{t-T}{\tau}\right) & \text{if } \; t > T \end{cases}$$

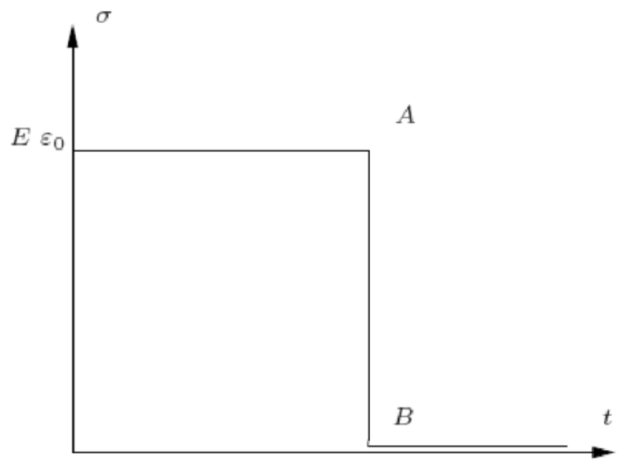




- Response to the relaxation test with loading and unloading:

$$\sigma(t) = \begin{cases} E.\epsilon_0 & \text{if } \; 0 \le t \le T \\ 0 & \text{if } \; t > T \end{cases}$$

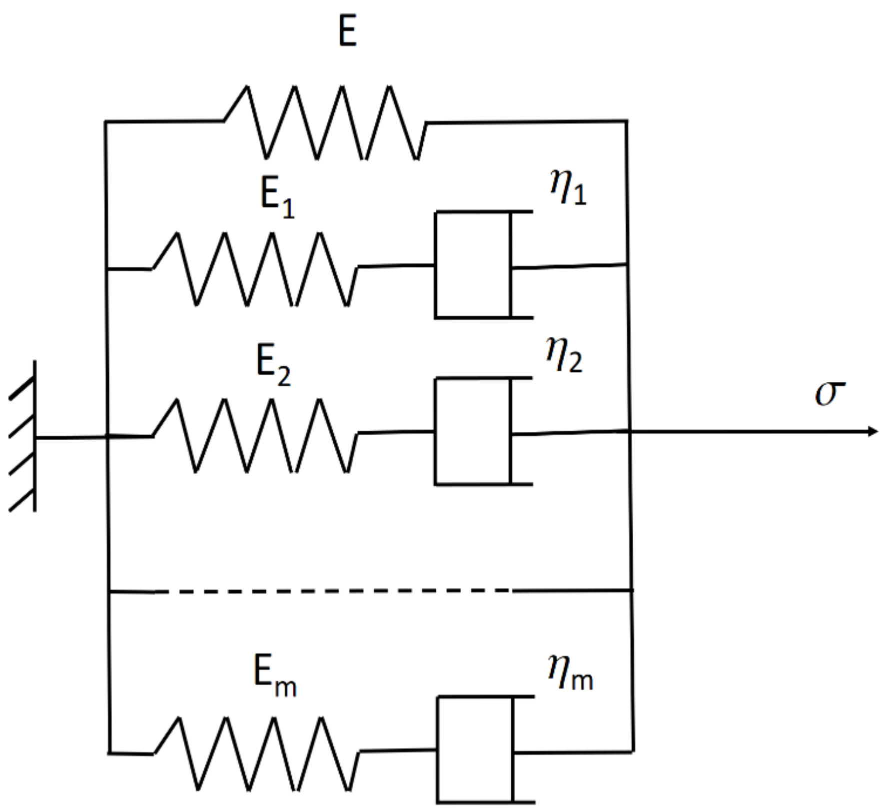




Time-dependent behavior: linear viscoelastic models

Generalized Maxwell model

- The generalized Maxwell model consists of a spring and *m* Maxwell models assembled in parallel



- Using this model for the relaxation test:

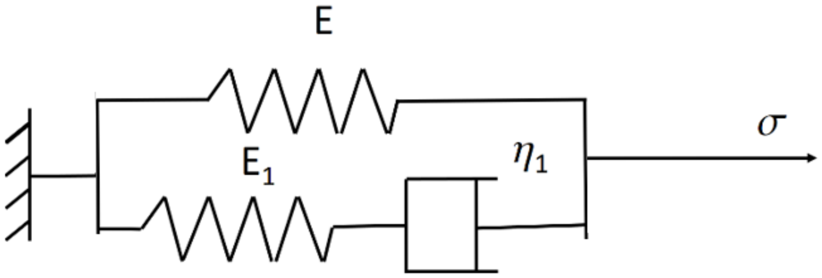
$$\sigma = \sigma^{E} + \sum_{i=1}^{m} \sigma^{M_i}$$

$$r_{LC} = r_{LC}^{E} + r_{LC}^{M_i} = E + \sum_{j=1}^{m} \frac{E_j s}{s + \frac{E_j}{\eta_j}}$$

$$f_{LC} = \frac{1}{r_{LC}} \xrightarrow{\text{Laplace inverse}} f(t), r(t)$$

Time-dependent behavior: linear viscoelastic models

Zener model



- Zener model (particular case of the generalized Maxwell model with m = 1) : is obtained by assembling Maxwell's model in parallel with a spring

- The relaxation function in temporal space and that of Laplace-Carson can be deduced by an addition of the component functions of the spring and those of the Maxwell model

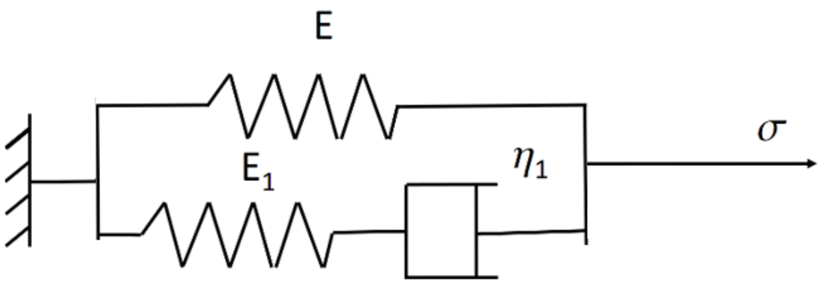
$$r_{LC} = r_{LC}^{E} + r_{LC}^{M_1} = E + \frac{E_1 s \eta_1}{\eta_1 s + E_1} \Rightarrow r(t) = E + E_1 e^{-\frac{E_1 t}{\eta_1}}$$

- Creep function:

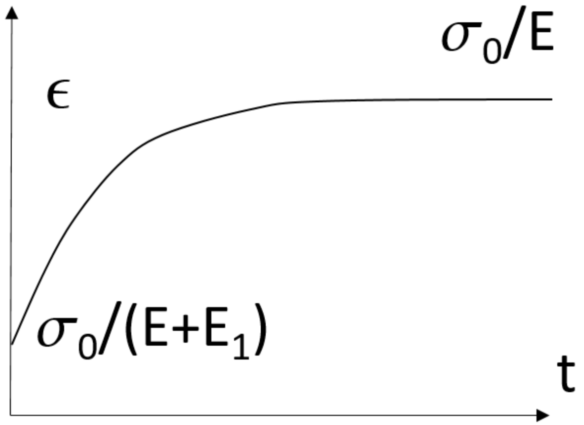
$$f_{LC} = \frac{1}{r_{LC}} = \frac{E_1 + \eta_1 s}{\eta_1 s (E + E_1) + E_1 E} \Rightarrow f(t) = \frac{1}{E}\left[1 - \frac{E_1}{E + E_1} e^{-\frac{E_1 E}{(E+E_1)\eta_1} t}\right]$$

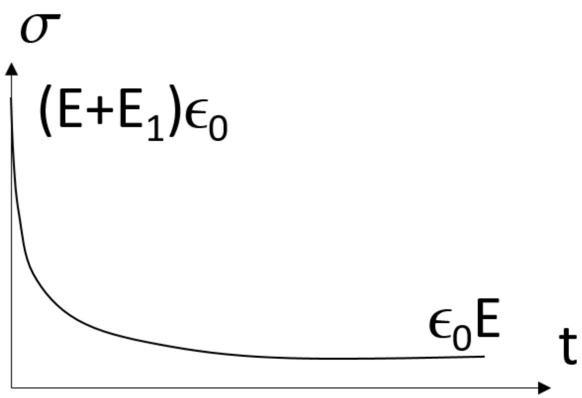
Time-dependent behavior: linear viscoelastic models

Zener model



- Response to creep and relaxation tests:
  - instataneous (t=0): $f(t)=1/(E+E_1) = 1/r(t)$
  - to infinity ($t \longrightarrow \infty$): $\epsilon(t)=\sigma_0/E$ (creep test) whilst $\sigma(t)=\epsilon_0.E$ (relaxation test).

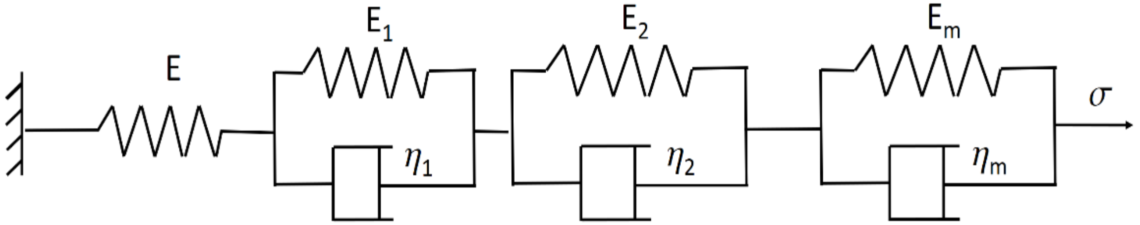




Time-dependent behavior: linear viscoelastic models

Generalized Kelvin model

- The generalized Kelvin model consists of a spring and *m* Kelvin models assembled in series



- Using this model for the creep test:

$$\varepsilon = \varepsilon^{E} + \sum_{i=1}^{m} \varepsilon^{K_i}$$

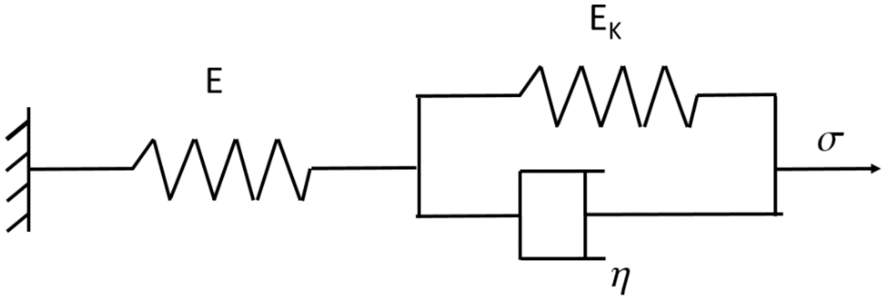
$$f_{LC} = f_{LC}^{E} + f_{LC}^{K_i} = \frac{1}{E} + \sum_{j=1}^{m} \left(\frac{1}{\eta_j s + E_j} \right)$$

$$r_{LC} = \frac{1}{f_{LC}} \xrightarrow{\text{Laplace inverse}} f(t), r(t)$$

Time-dependent behavior: linear viscoelastic models

Kelvin-Voigt model

- Kelvin-Voigt model (particular case of the generalized Kelvin model with *m* = 1) : is obtained by assembling the Kelvin model in series with a spring.



- The creep function in temporal space and that of Laplace-Carson can be deduced by an addition of the component functions of the spring and those of the Kelvin model

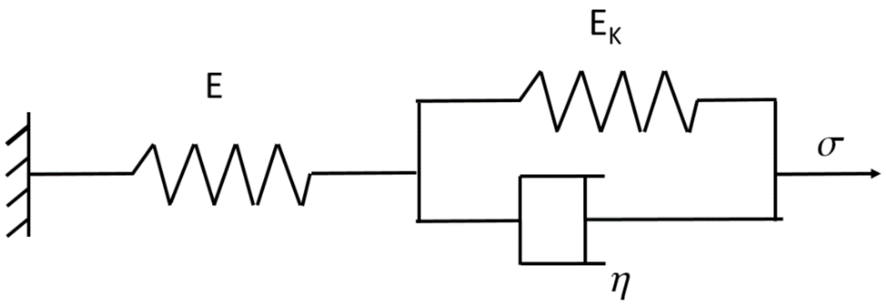
$$f(t) = \frac{1}{E} + \frac{1}{E_K}\left(1 - e^{-\frac{E_K}{\eta}t}\right), \quad f_{LC} = \frac{1}{E} + \frac{1}{E_K + \eta.s}$$

- Relaxation function:

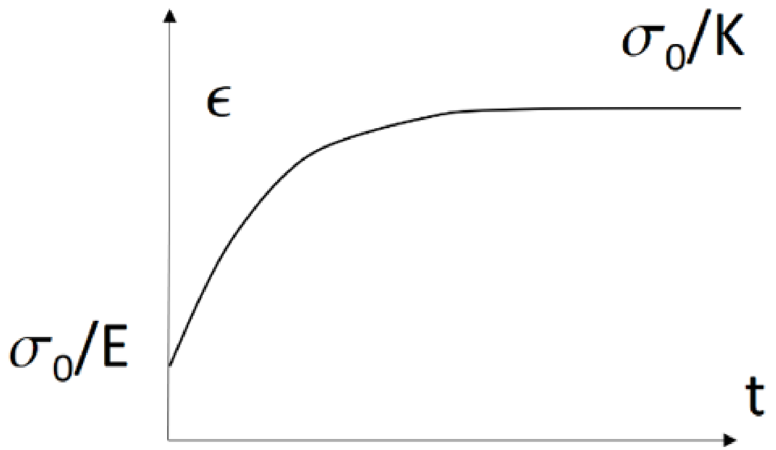
$$r_{LC} = \frac{1}{f_{LC}} = E\,\frac{E_K + \eta.s}{E + E_K + \eta.s} \quad \Rightarrow \quad r(t) = \left[K + (E - K)e^{-\frac{(E+E_K).t}{\eta}}\right]Y(t) \quad \text{with} \quad K = \frac{E.E_K}{E + E_K}$$

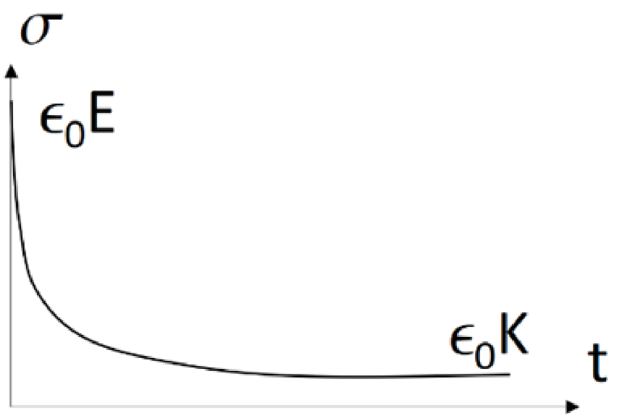
Time-dependent behavior: linear viscoelastic models

Kelvin-Voigt model



- Response to creep and relaxation tests:
  - instataneous (t=0): f(t)=E=1/r(t)
  - to infinity ($t \longrightarrow \infty$): $\epsilon(t)=\sigma_0/K$ (creep test) whilst $\sigma(t)=\epsilon_0.K$ (relaxation test).

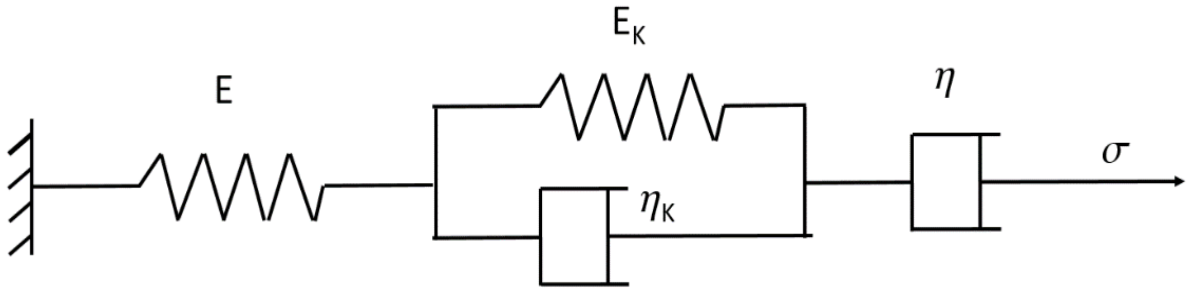




Time-dependent behavior: linear viscoelastic models

Burger model

- The Burger viscoelastic model is obtained by assembling the Kelvin model in series with the Maxwell model.



- The creep function in temporal space and that of Laplace-Carson can be deduced by an addition of the component functions of the Maxwell model and those of the Kelvin model:

$$f(t)=\left(\frac{1}{E}+\frac{1}{E_K}(1-e^{-\frac{E_K}{\eta_K}t})+\frac{t}{\eta}\right)Y(t),\quad f_{LC}=\frac{1}{E}+\frac{1}{E_K+\eta_K.s}+\frac{1}{\eta.s}$$

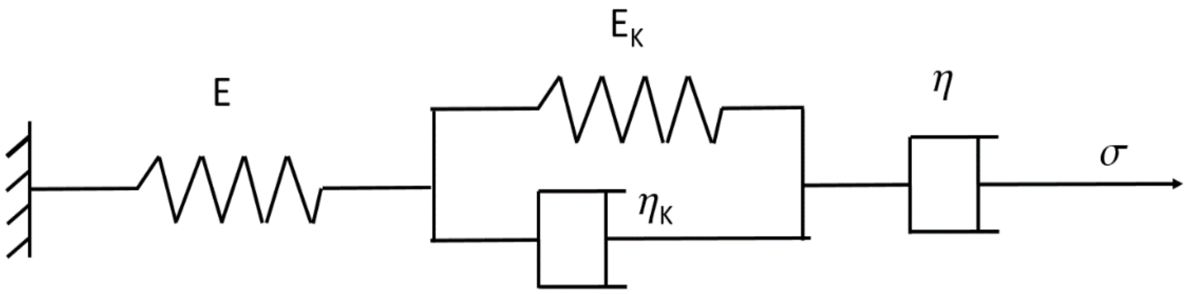
- Relaxation function:

$$r_{LC}=\frac{1}{f_{LC}}=\frac{E.s(E_K+\eta_K.s)}{E.s+(E_K+\eta_K.s)(E/\eta+s)}$$

$$\begin{cases} r(t)=\dfrac{E}{2.a}(a.e^{-\frac{t(b-a)}{d}}-c.e^{-\frac{t(b-a)}{d}}+a.e^{-\frac{t(b+a)}{d}}+c.e^{-\frac{t(b+a)}{d}}) \\ a=\sqrt{(E_K\eta+E(\eta+\eta_K))^2-4E_KE\eta_K\eta};\quad b=E(\eta_K+\eta)+E_K\eta; \\ c=E(\eta_K+\eta)-E_K\eta;\quad d=2\eta_K\eta; \end{cases}$$

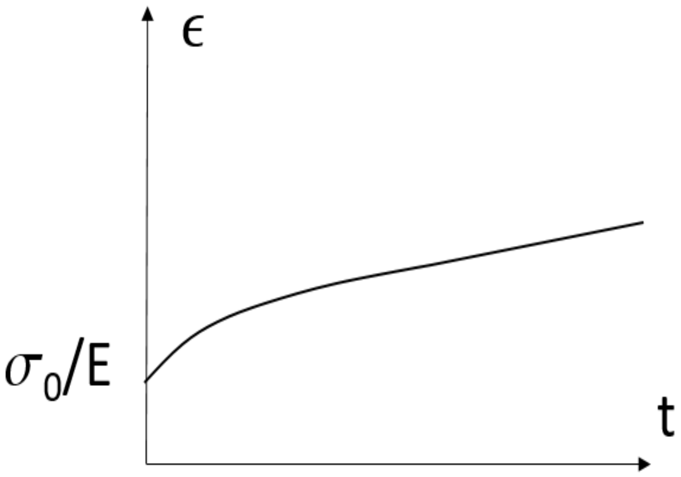
Time-dependent behavior: linear viscoelastic models

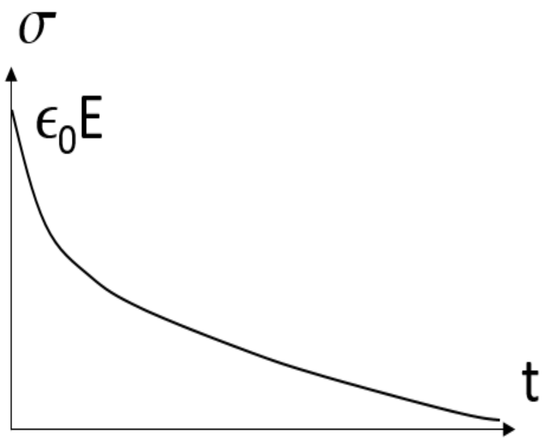
Burger model



- Response to creep and relaxation tests:
  - instataneous (t=0): *f(0)=1/E* et *r(0)=E*

to infinity ($t \longrightarrow \infty$): *f(t)* $\longrightarrow \infty$ (creep test) whilst *r(t)* $\longrightarrow 0$ (relaxation test).





Time-dependent behavior: linear viscoelastic models

Application of the correspondence principle: examples

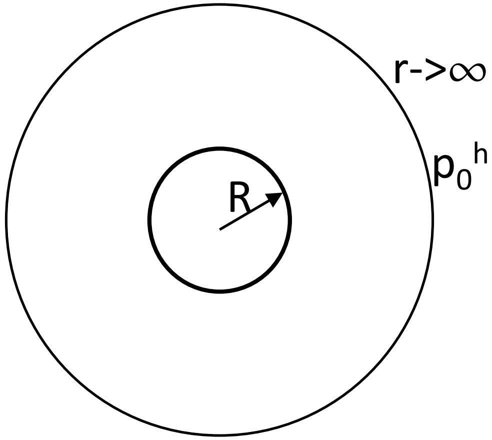
- **Example 1:** convergence of a circular and unsupported tunnel in a linear and incompressible viscoelastic rock mass under the state of hydrostatic stress

- Equilibrium equation:

$$div\sigma = 0 \quad \Rightarrow \quad \frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

- Boundary conditions:

$$\sigma_r(r = R) = 0, \quad \sigma_r(r \to \infty) = p_0^h$$



- Compatibility conditions:

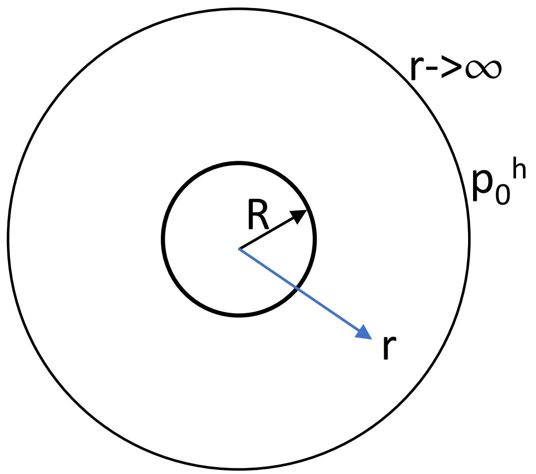
$$\varepsilon_r = \frac{\partial u}{\partial r}, \quad \varepsilon_\theta = \frac{u}{r}$$

Time-dependent behavior: linear viscoelastic models

- ***Case of linear elastic rock:***

$$\sigma_r = \frac{E}{1+\nu}\left(\frac{1-\nu}{1-2\nu}\varepsilon_r + \frac{\nu}{1-2\nu}\varepsilon_\theta\right)$$

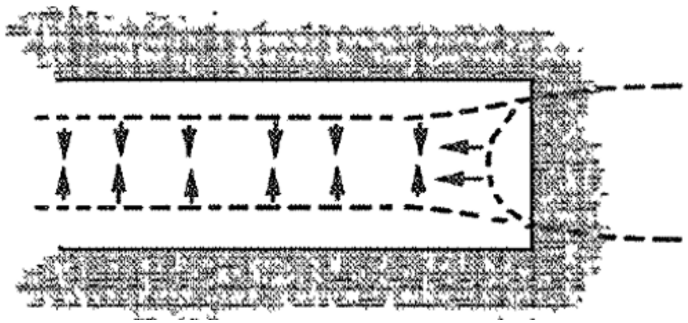
$$\sigma_\theta = \frac{E}{1+\nu}\left(\frac{1-\nu}{1-2\nu}\varepsilon_\theta + \frac{\nu}{1-2\nu}\varepsilon_r\right)$$



- Classical solution of the isotropic problem and under hydrostatic load:

$$u = a.r + \frac{b}{r}$$

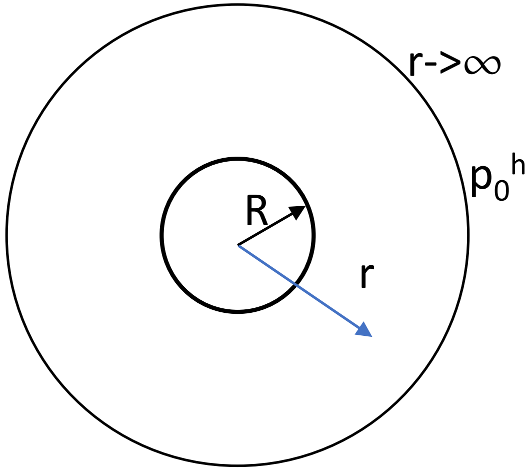
$$\varepsilon_r = \frac{\partial u}{\partial r} = a - \frac{b}{r^2}, \quad \varepsilon_\theta = a + \frac{b}{r^2}$$



Time-dependent behavior: linear viscoelastic models

$$\sigma_r = \frac{E}{(1+\nu)(1-2\nu)}\left((1-\nu)\left(a-\frac{b}{r^2}\right)+\nu\left(a+\frac{b}{r^2}\right)\right)$$

$$\sigma_\theta = \frac{E}{(1+\nu)(1-2\nu)}\left((1-\nu)\left(a+\frac{b}{r^2}\right)+\nu\left(a-\frac{b}{r^2}\right)\right)$$



- Replacing the boundary conditions in the equation of radial stress:

$$\begin{cases} \sigma_r(r=R) = \dfrac{E\left(aR^2 - b(1-2\nu)\right)}{(1+\nu)(1-2\nu)} = 0 \\ \sigma_r(r\to\infty) = \dfrac{aE}{(1+\nu)(1-2\nu)} = p_0^h \end{cases} \Rightarrow \begin{cases} a = \dfrac{(1+\nu)(1-2\nu)}{E}p_0^h \\ b = \dfrac{(1+\nu)R^2}{E}p_0^h \end{cases}$$

Time-dependent behavior: linear viscoelastic models

$$u = a.r + \frac{b}{r} = \frac{(1+\nu)(1-2\nu)\,p_0^h}{E} r + \frac{1+\nu}{E}\frac{R^2 p_0^h}{r} = \frac{p_0^h}{3K} r + \frac{R^2 p_0^h}{2G.r}$$

$$K = \frac{E}{3(1+\nu)(1-2\nu)}, \quad G = \frac{E}{2(1+\nu)}$$

- Case of incompresible rock: $\nu = 0.5, \quad K \to \infty$

$$u = \frac{p_0^h}{3K} r + \frac{R^2 p_0^h}{2G.r}$$

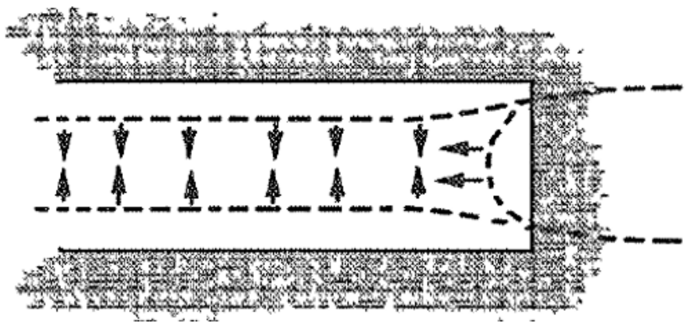
$$u_R^{elast}(r = R) = \frac{R p_0^h}{2G}$$

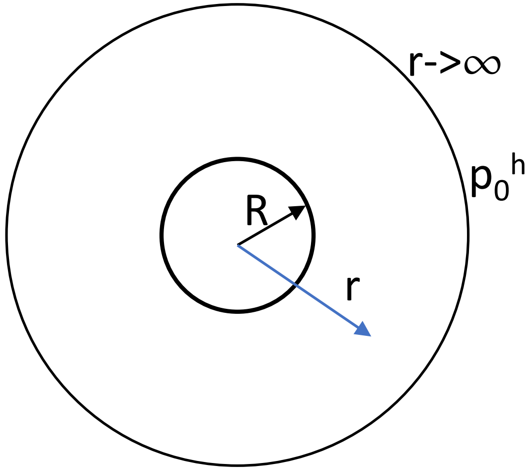
- Stress state: time-independent

$$\begin{cases} \sigma_r = (1 - \dfrac{R^2}{r^2}) p_0^h \\ \sigma_\theta = (1 + \dfrac{R^2}{r^2}) p_0^h \end{cases}$$

Time-dependent behavior: linear viscoelastic models

$$\begin{cases} \sigma_r = (1 - \frac{R^2}{r^2}) p_0^h \\ \sigma_\theta = (1 + \frac{R^2}{r^2}) p_0^h \end{cases}$$





- The stress state in the rock mass is constant versus time: in the case of the linear viscoelastic rock we can **apply the correspondence principle (the problem is equivalent to the problem of creep)**

$$u_R^{elast} = \frac{R p_0^h}{2G} \quad \Rightarrow \quad u_{LC} = \frac{R p_0^h}{2} \frac{1}{G_{LC}}$$

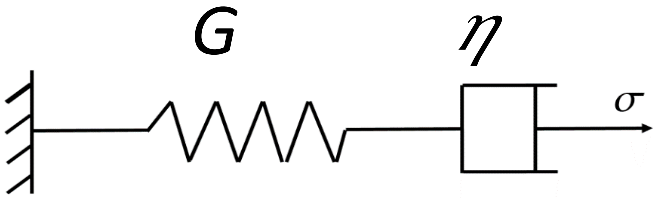
Time-dependent behavior: linear viscoelastic models

- *Case of Maxwell model of rock*

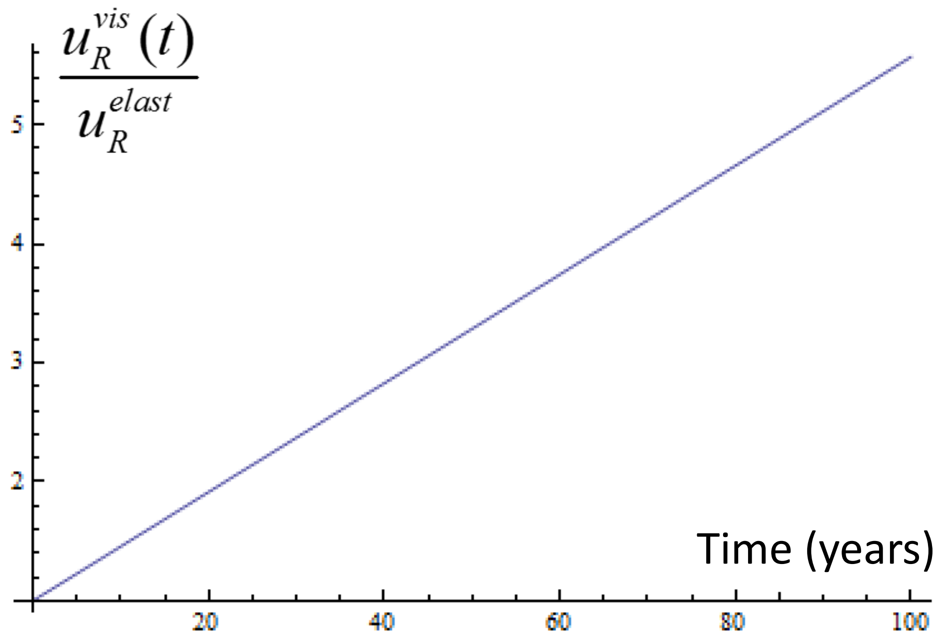
$$u_R^{elast} = \frac{Rp_0^h}{2G} \qquad G_{LC} = \frac{G.s.\eta}{\eta s + G}$$

$$u_{LC} = \frac{Rp_0^h}{2} \frac{G + s.\eta}{G.s.\eta}$$

$$u_R^{vis}(t) = \frac{Rp_0^h}{2}\left(\frac{1}{G} + \frac{t}{\eta}\right)$$



- Numerical application: G=6.0(GPa), η=4.14x10^9(GPa/s)



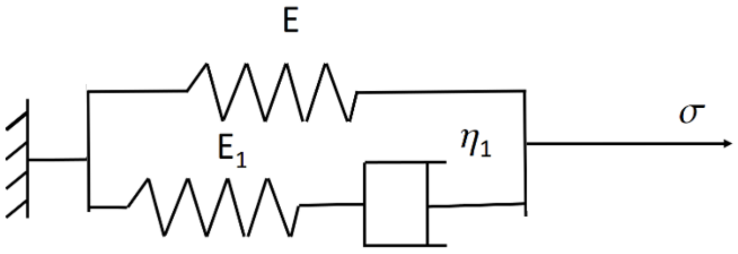
Time-dependent behavior: linear viscoelastic models

- *Case of Kelvin model of rock*

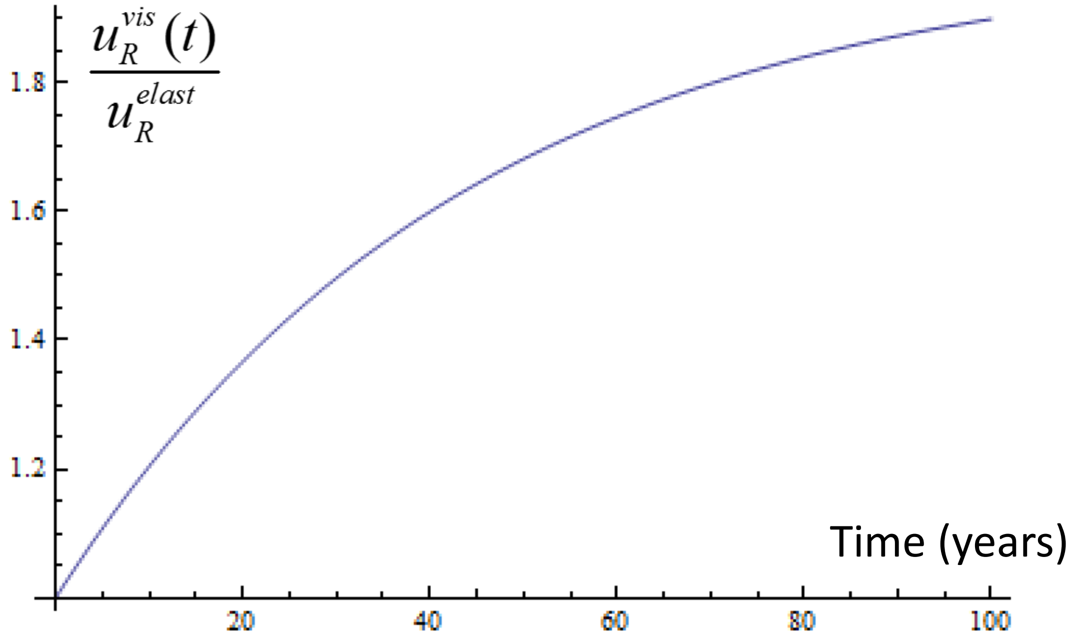
$$u_R^{elast} = \frac{Rp_0^h}{2G} \qquad G_{LC} = G + \frac{G_1\eta_1 s}{\eta_1 s + G_1}$$

$$u_{LC} = \frac{Rp_0^h}{2}\frac{1}{\left(G + \frac{G_1\eta_1 s}{\eta_1 s + G_1}\right)}$$

$$\Rightarrow \frac{u_R^{vis}(t)}{u_R^{elast}} = \left(\frac{G+G_1}{G} - \frac{G_1 e^{-\frac{G_1 G t}{(G+G_1)\eta_1}}}{G}\right)$$



- Numerical application: $G=G_1=6.0(GPa)$, $\eta=4.14\times10^9(GPa/s)$



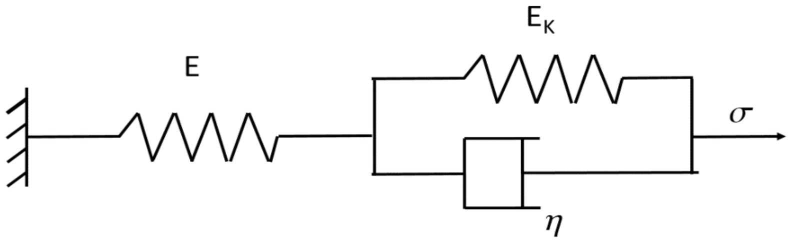
Time-dependent behavior: linear viscoelastic models

- *Case of Kelvin-Voigt model of rock*

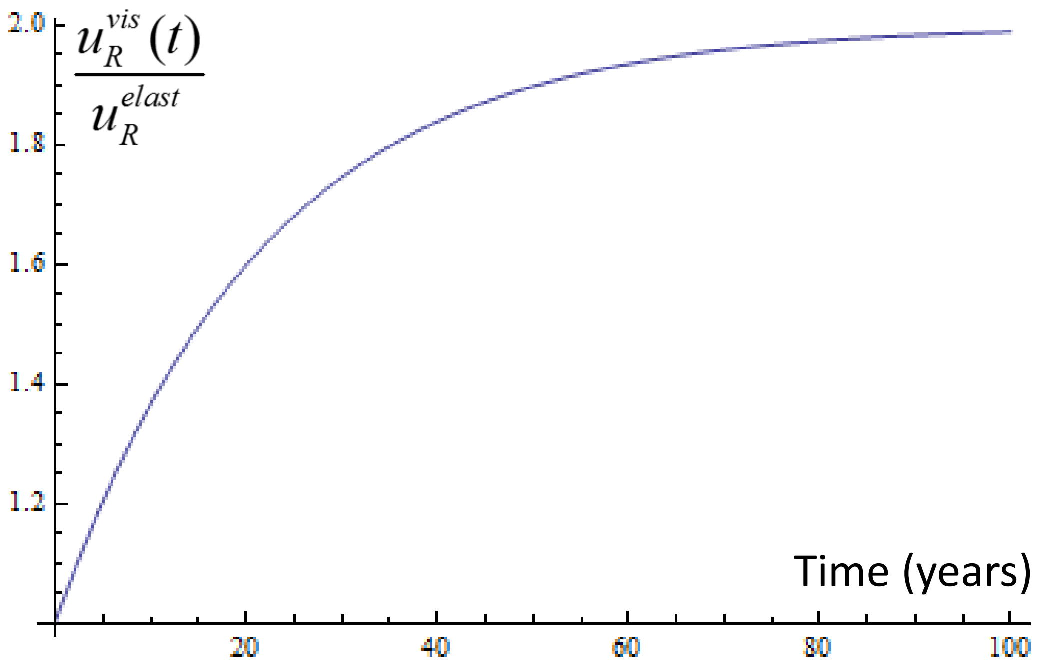
$$u_R^{elast} = \frac{Rp_0^h}{2G} \qquad G_{LC} = G\frac{G_K + \eta s}{\eta s + G_K + G}$$

$$u_{LC} = \frac{Rp_0^h}{2}\frac{\left(G + G_K + \eta s\right)}{G\left(G_K + \eta s\right)}$$

$$\frac{u_R^{vis}(t)}{u_R^{elast}} = \left(\frac{G + G_K}{G_K} - \frac{Ge^{-\frac{G_K t}{\eta}}}{G_K}\right)$$



- Numerical application: G=G$_K$=6.0(GPa),η=4.14x10^9(GPa/s)



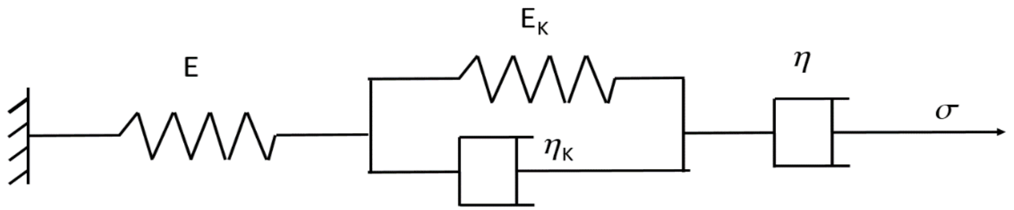
Time-dependent behavior: linear viscoelastic models

- *Case of Burger model of rock*

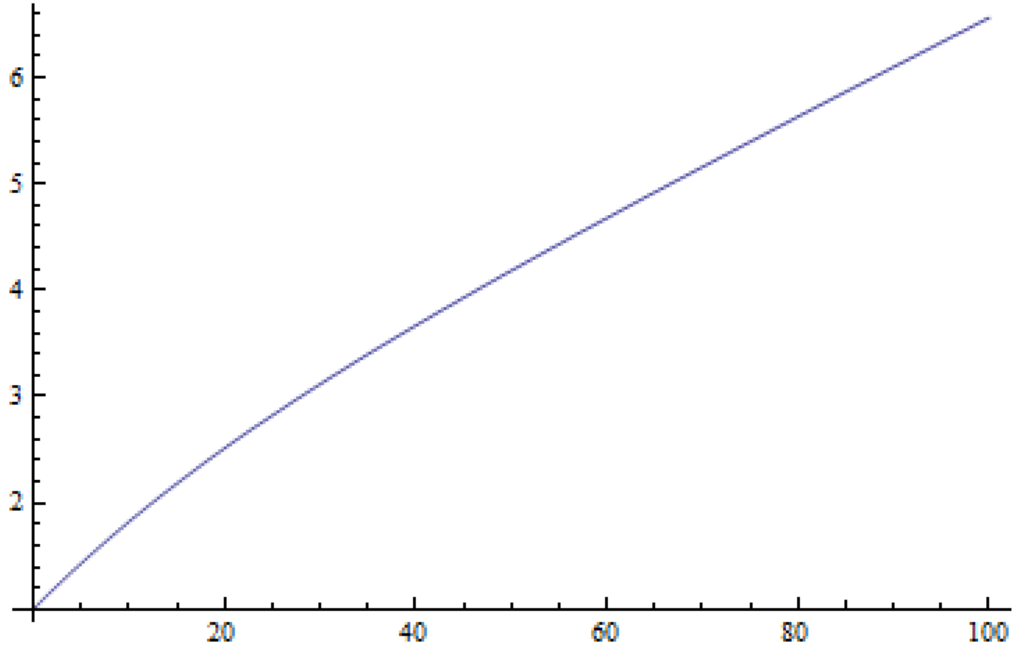
$$G_{LC} = \frac{Gs(G_K + \eta_K s)}{Gs + (\eta_K s + G_K)(G/\eta + s)}$$

$$u_{LC} = \frac{Rp_0^h}{2} \frac{\left(Gs + \left(G_K + \eta_K s\right)\left(G/\eta + s\right)\right)}{Gs\left(G_K + \eta_K s\right)}$$

$$\frac{u_R^{vis}(t)}{u_R^{elast}} = 1 + \frac{G\left(G_K t + \eta - \eta e^{-\frac{G_K t}{\eta_K}}\right)}{G_K \eta}$$

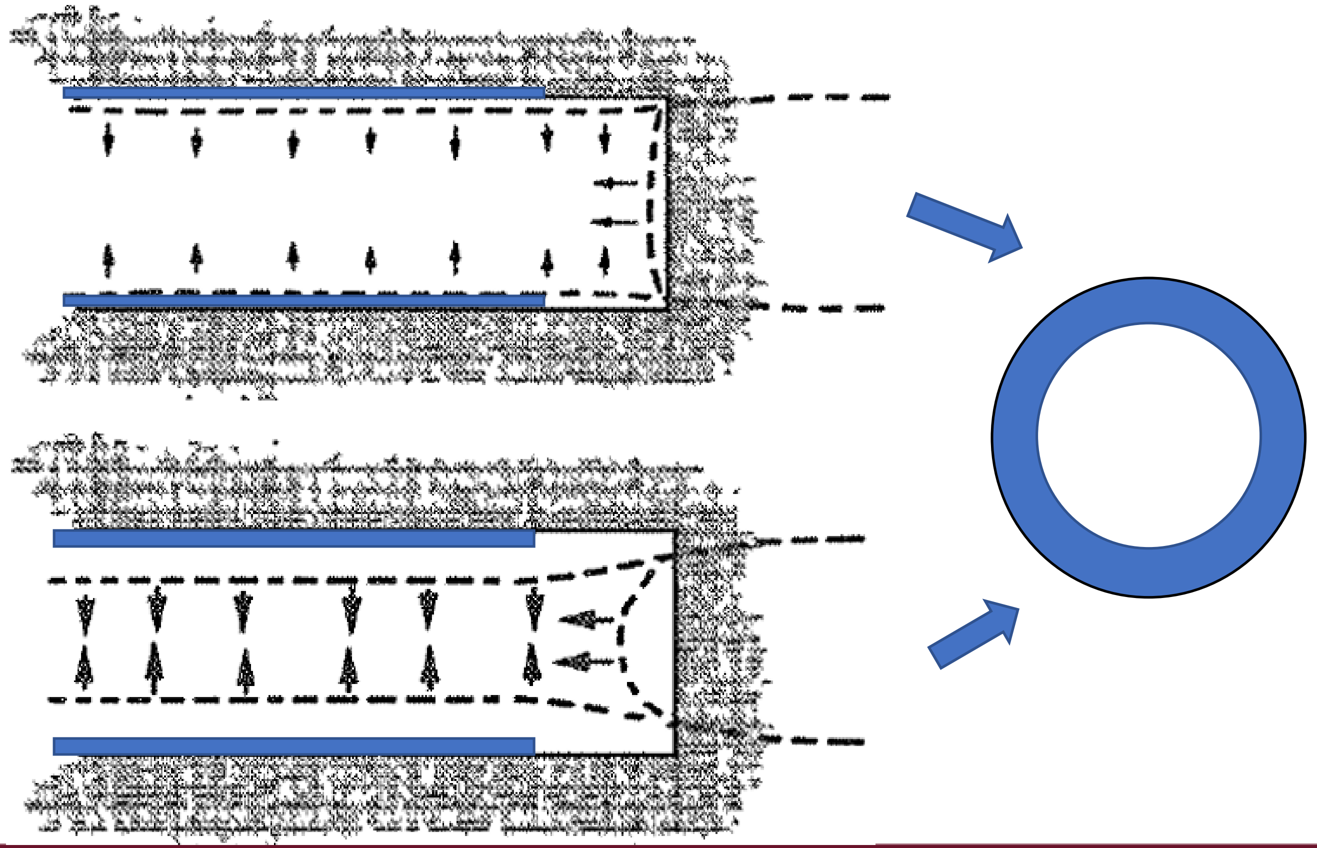


- Numerical application: $G=G_K=6.0$(GPa),$\eta=\eta_K=4.14\text{x}10^9$(GPa/s)



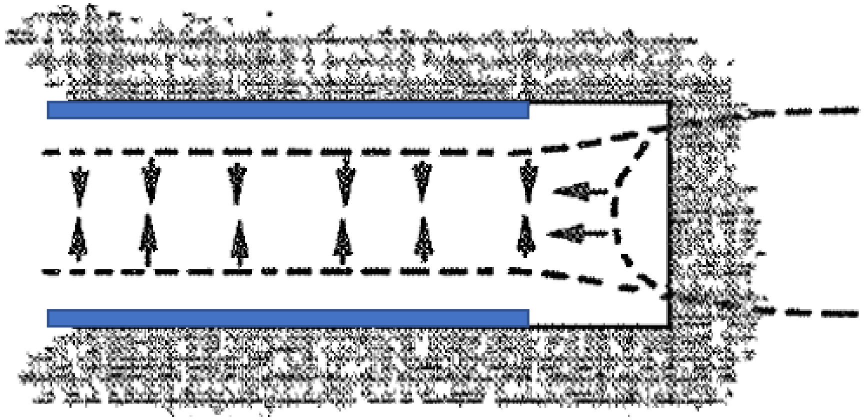
Time-dependent behavior: linear viscoelastic models

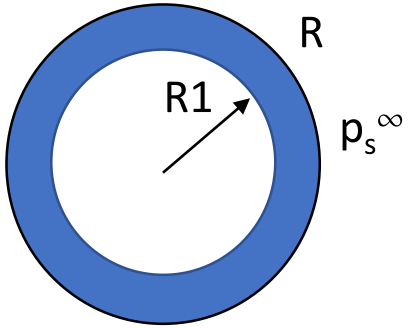
- Influence of the creep behavior of the rock mass on the design of the liner

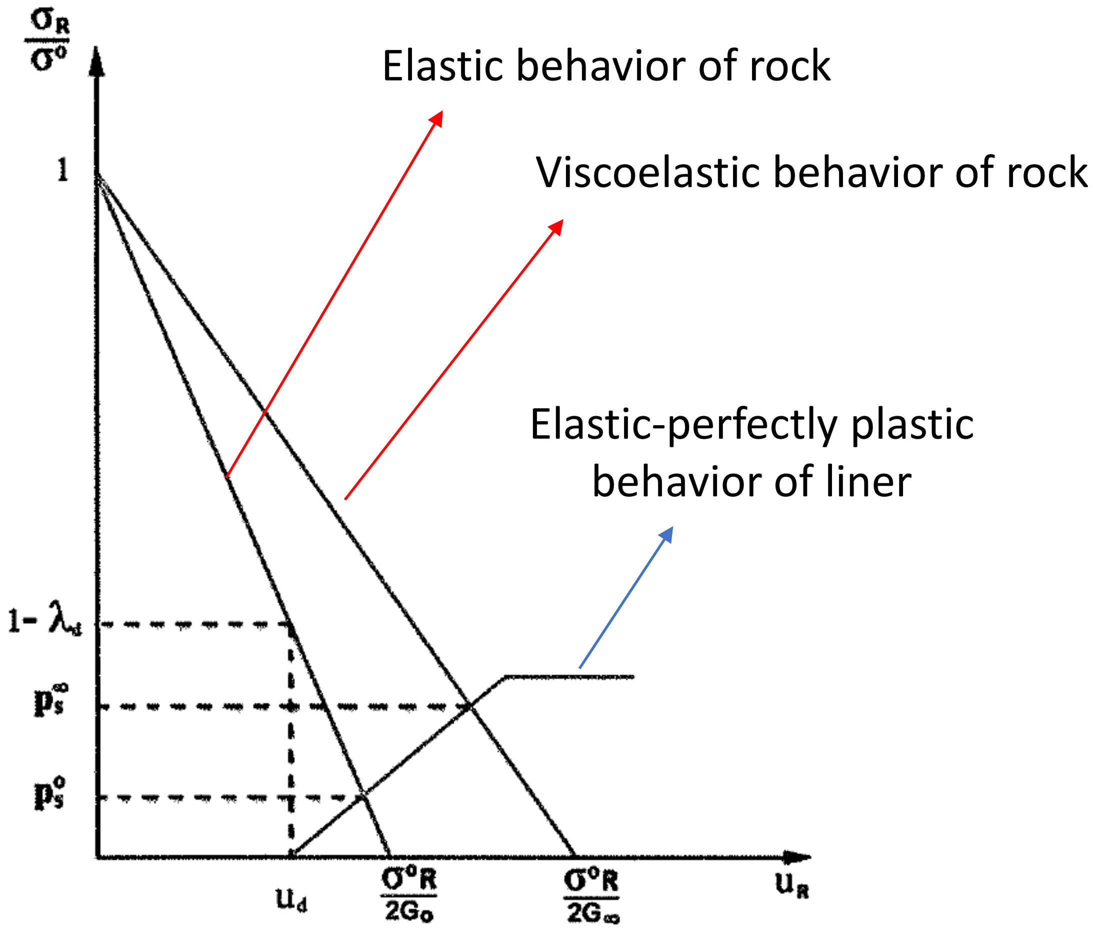


Time-dependent behavior: linear viscoelastic models

- Influence of the creep behavior of the rock mass on the design of the liner



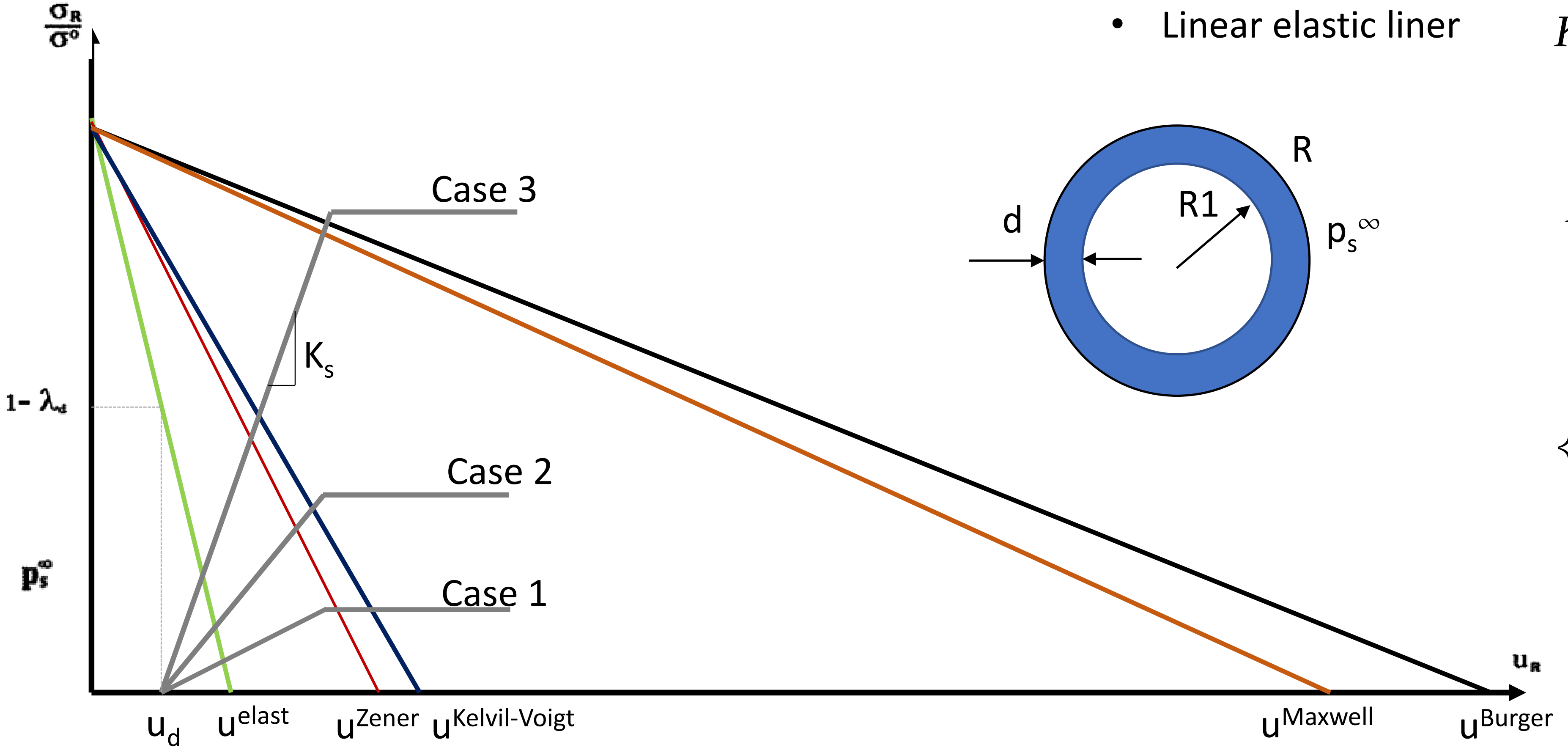




Convergence-confinement method

Time-dependent behavior: linear viscoelastic models

- Influence of the creep behavior of the rheological rock mass



- Linear elastic liner

$$K_s = \frac{R^2 - R_1^2}{\frac{(1+\nu_L)R^2}{K_L} + \frac{R_1^2}{2G_L}},$$

$$R_1 = R - d,$$

$$\begin{cases} \sigma_r = (1 - \frac{R_1^2}{r^2})p_s^\infty \\ \sigma_\theta = (1 + \frac{R_1^2}{r^2})p_s^\infty \end{cases}$$

Time-dependent behavior: linear viscoelastic models

Application of the correspondence principle: examples

❑ **Example 2**: ***creep behavior induces loss of prestress***

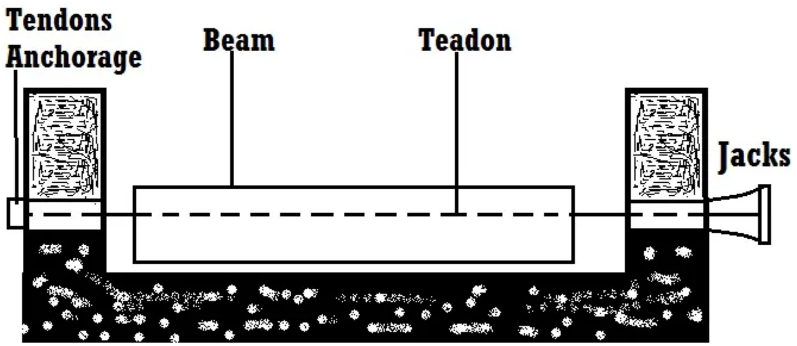
- Considering a viscoelastic element which is prestressed by the elastic reinforcements with the tensile stress σ_0 at the initial time instant t_0

  - By noting the stress in the reinforcements $\sigma_1(t)$ and in the concrete $\sigma_2(t)$.

$$\sigma_1(t_0) = \sigma_0, \quad \sigma_2(t_0) = -\sigma_0$$

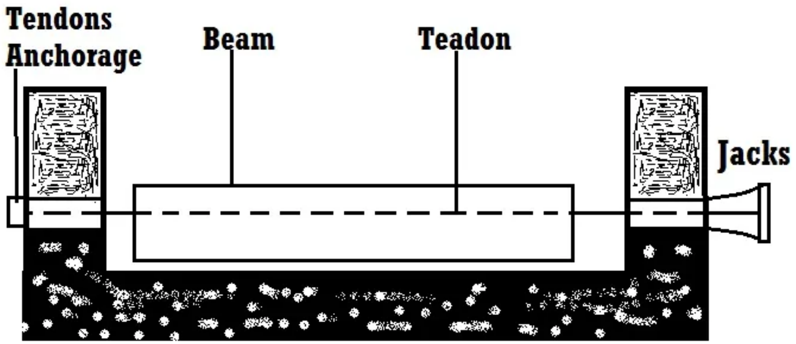
- Instantaneous behavior ($t=t_0$): $$\varepsilon_1(t_0) = \frac{\sigma_0}{E_s}, \quad \varepsilon_2(t_0) = -f(t_0)\sigma_0$$

E_s: elastic modulus of reinforcement
$f(t)$: creep function of concrete

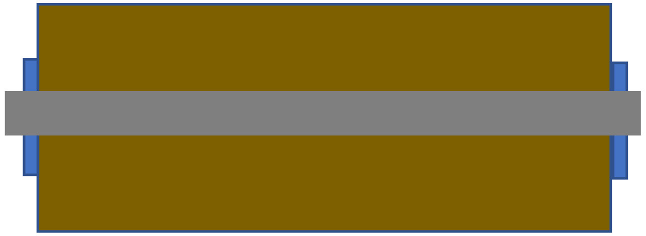


Pretension system

Time-dependent behavior: linear viscoelastic models



Pretension system



- As a function of time: the viscoelastic behavior of concrete induces a compressive strain (in the concrete and also in the reinforcement) and hence reduces the stress in each material (concrete, steel)

- Without external loading of the system:

$$\sigma_1(t)+\sigma_2(t)=0$$

$$\sigma_2(t)=-\sigma_1(t)$$

- the bond between concrete and steel conditions the same shortening in these two materials:

$$\varepsilon_1(t)-\varepsilon_1(t_0)=\varepsilon_2(t)-\varepsilon_2(t_0)$$

- Using the stress-strain relationship in the elastic and viscoelastic behavior of reinforcement and concrete:

$$\frac{\sigma_1(t)-\sigma_1(t_0)}{E_s}=f\,'{*}\,\sigma_2-f(t_0)\sigma_2(t_0)$$

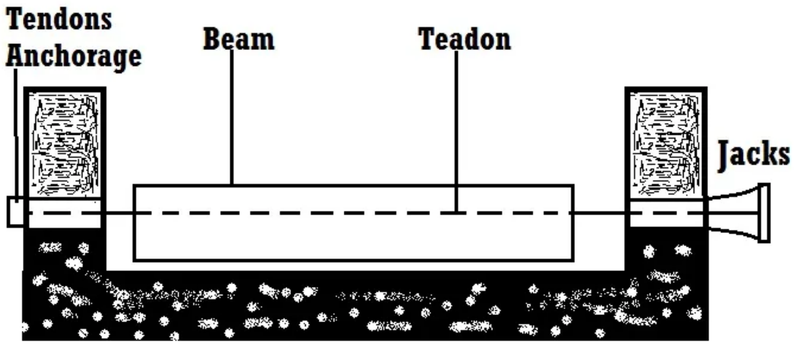
Time-dependent behavior: linear viscoelastic models

- From the conditions:

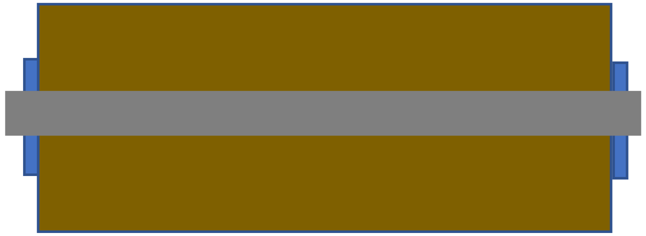
$$\sigma_2(t_0) = -\sigma_1(t_0) = -\sigma_0$$
$$\sigma_2(t) = -\sigma_1(t)$$

$$\frac{\sigma_1(t) - \sigma_1(t_0)}{E_s} = f\,'{*}\sigma_2 - f(t_0)\sigma_2(t_0)$$

$$\left(f(t) + \frac{1}{E_s}\right)'{*}\sigma_2 = -\left(\frac{1}{E_s} + f(t_0)\right)\sigma_0$$



Pretension system



- The stress state in the concrete element can be written in form:

$$g\,'{*}\sigma_2 = -g(t_0)\sigma_0, \quad g(t) = f(t) + \frac{1}{E_s}$$

Laplace-Carson transform:

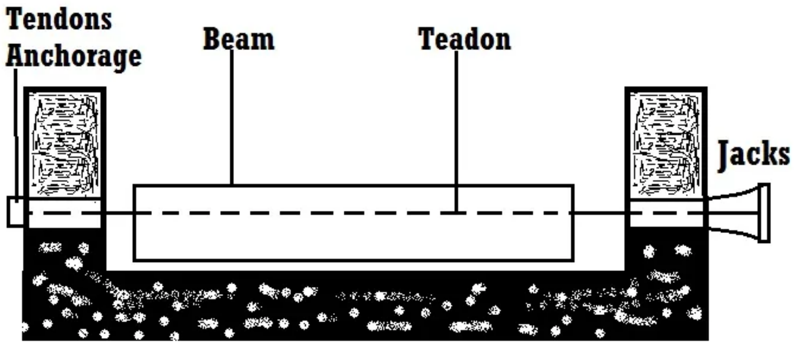
$$g_{LC}\left(\sigma_2\right)_{LC} = -g(t_0)\sigma_0$$

Time-dependent behavior: linear viscoelastic models

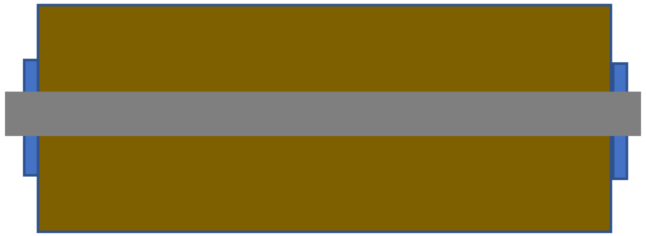
$$g_{LC}\left(\sigma_2\right)_{LC} = -g(t_0)\sigma_0$$

$$\left(\sigma_2\right)_{LC} = -g(t_0)\sigma_0 R_{LC}, \quad R_{LC}\, g_{LC} = 1$$

$$\sigma_2(t) = -\sigma_0 \frac{R(t)}{R(t_0)}$$



Pretension system

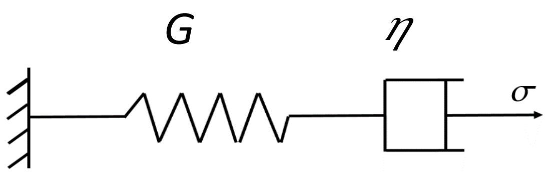


- Reduced of prestressed σ_2(t) in the concrete as a function of time:

$$\left|\frac{\sigma_2(t)}{\sigma_0}\right| = \left|\frac{R(t)}{R(t_0)}\right|$$

Time-dependent behavior: linear viscoelastic models

- *Case of Maxwell model of concrete*



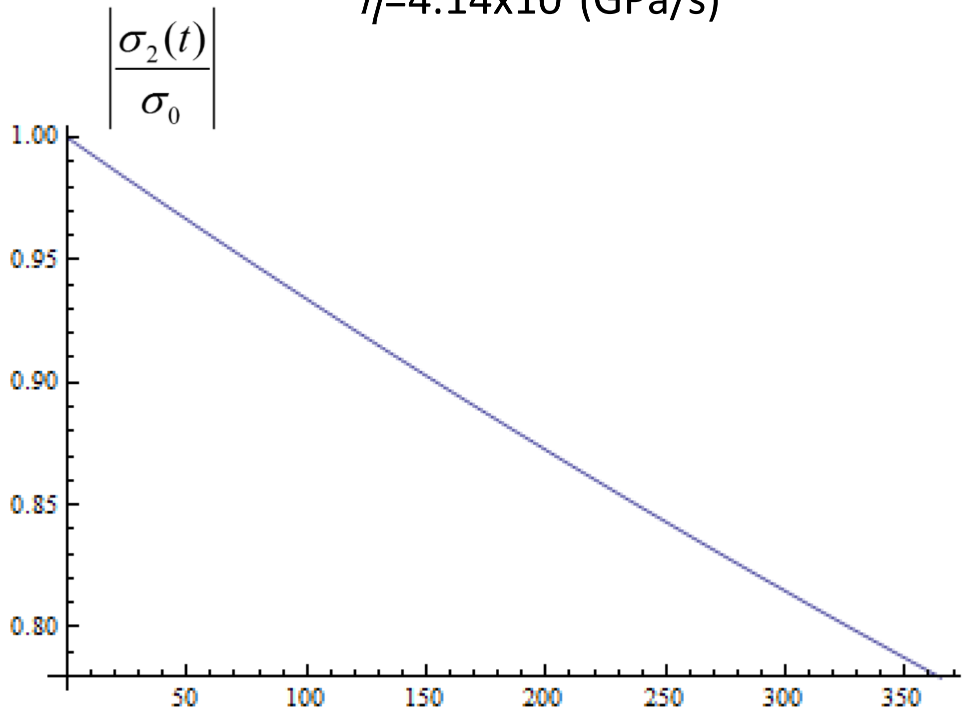
$$f(t)=\frac{1}{E}+\frac{t}{\eta} \Rightarrow g(t)=f(t)+\frac{1}{E_s}=\frac{1}{E}+\frac{1}{E_s}+\frac{t}{\eta}$$

E_s=200(GPa), E=39.1(GPa), η=4.14x10^9(GPa/s)

$$g_{LC}=\frac{1}{E}+\frac{1}{E_s}+\frac{1}{\eta s} \Rightarrow R_{LC}=\frac{1}{g_{LC}}$$

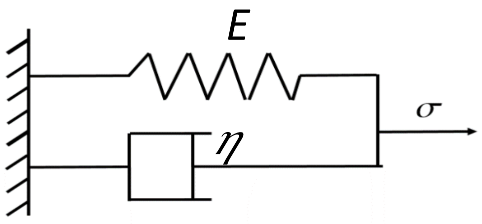
$$R(t)=\frac{E_s E}{E_s+E}e^{-\frac{E_s E}{E_s+E}\frac{t}{\eta}}$$

$$\Rightarrow \left|\frac{\sigma_2(t)}{\sigma_0}\right|=\left|\frac{R(t)}{R(t_0)}\right|=e^{-\frac{E_s E}{E_s+E}\frac{t}{\eta}}$$



Time-dependent behavior: linear viscoelastic models

- *Case of Kelvin model of concrete*



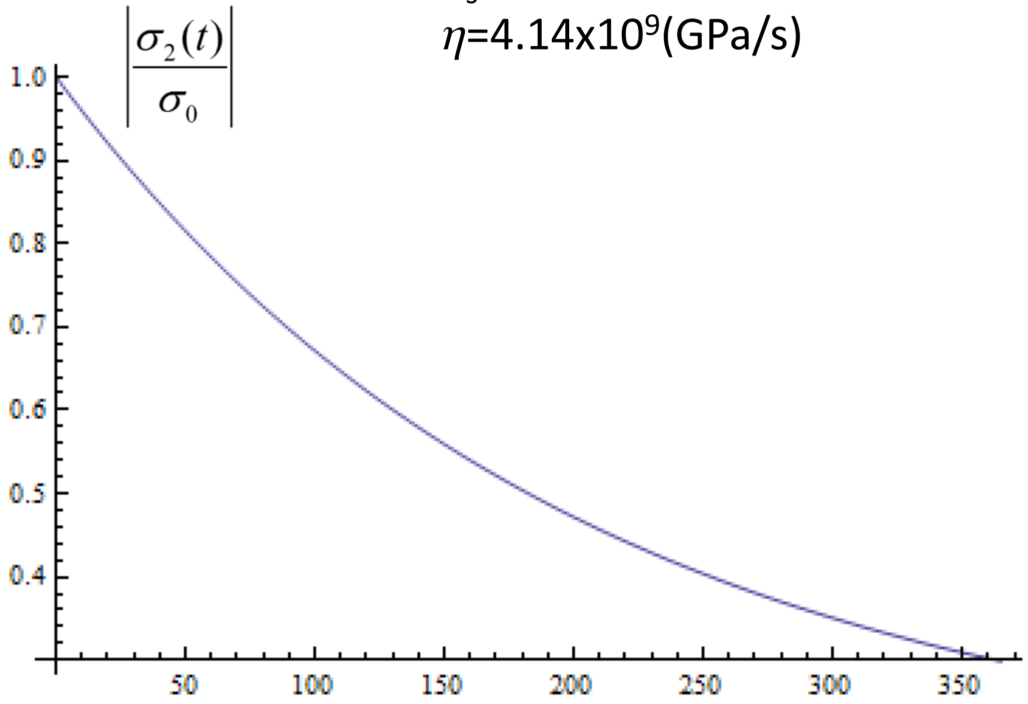
$$f(t)=\frac{1}{E}\left(1-e^{-\frac{E}{\eta}t}\right) \Rightarrow g(t)=f(t)+\frac{1}{E_s}=\frac{1}{E}\left(1-e^{-\frac{E}{\eta}t}\right)+\frac{1}{E_s}$$

$$g_{LC}=\frac{1}{E+\eta s}+\frac{1}{E_s} \Rightarrow R_{LC}=\frac{1}{g_{LC}}$$

$$R(t)=\frac{E_sE}{E_s+E}+\frac{E_s^2}{E_s+E}e^{-\frac{E_s+E}{\eta}t}$$

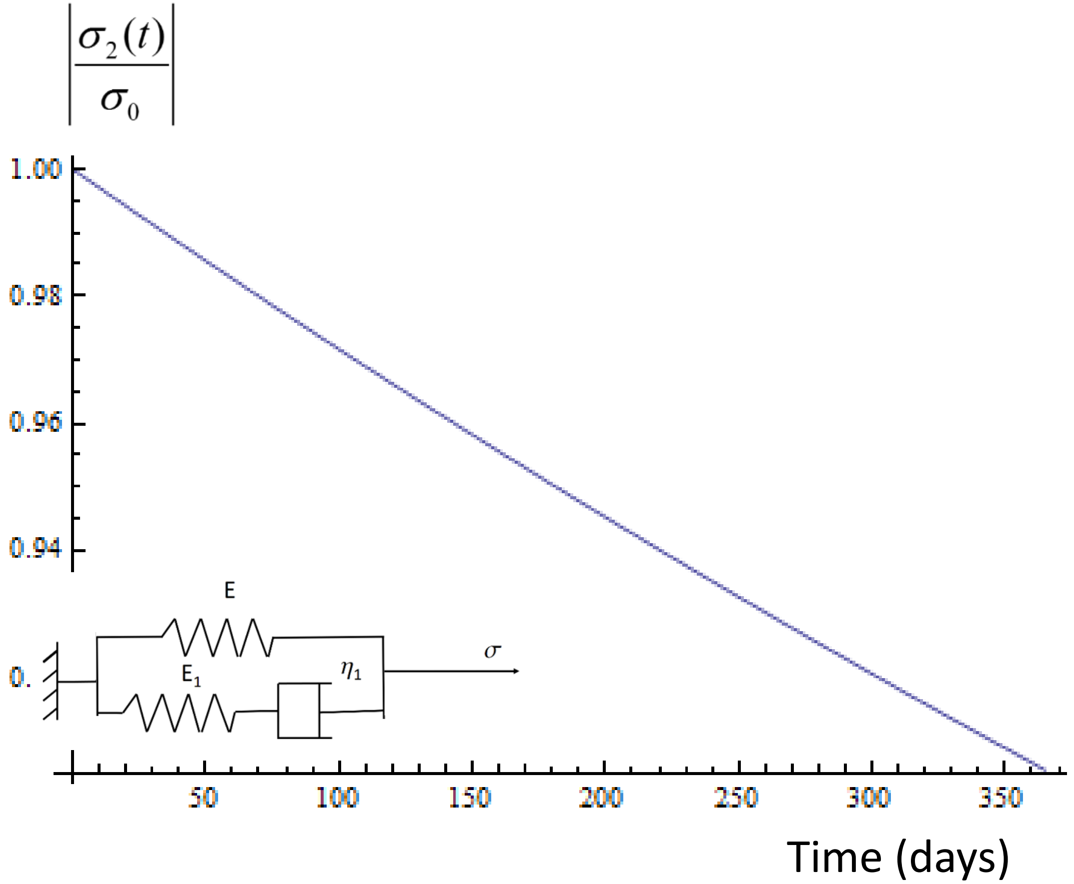
$$\Rightarrow \left|\frac{\sigma_2(t)}{\sigma_0}\right|=\left|\frac{R(t)}{R(t_0)}\right|=\frac{E+E_se^{-\frac{E_s+E}{\eta}t}}{E+E_s}$$

E_s=200(GPa), E=39.1(GPa), η=4.14x10^9(GPa/s)

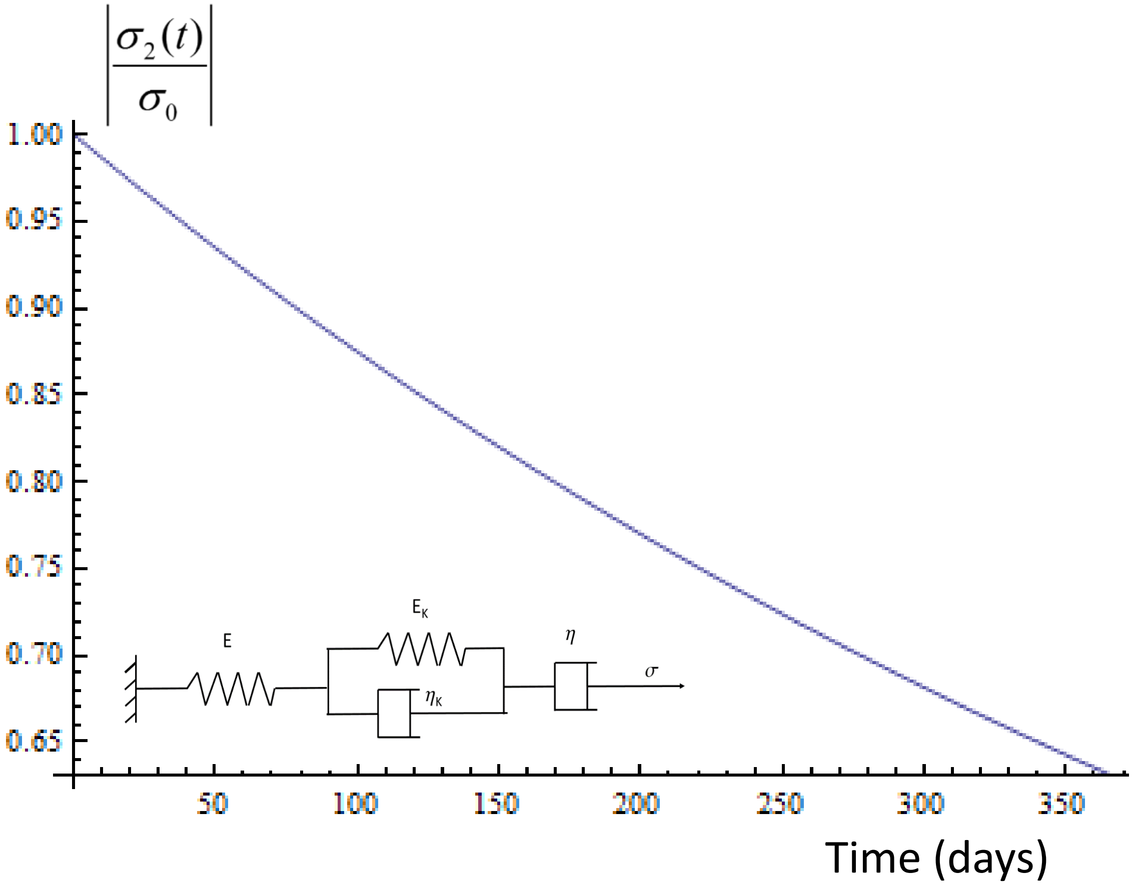


Time-dependent behavior: linear viscoelastic models

- *Case of Zener model of concrete*



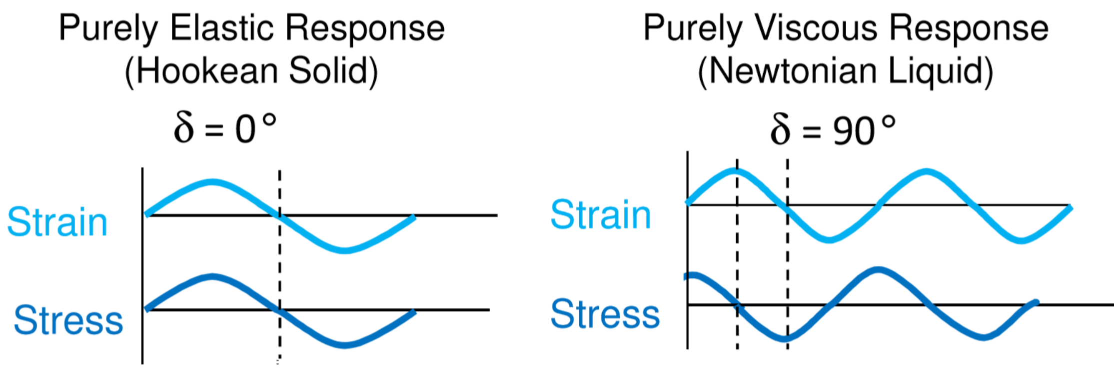
- *Case of Burger model of concrete*



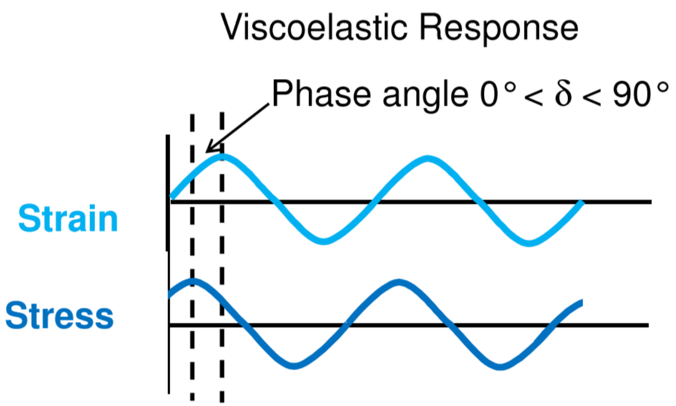
Time-dependent behavior: linear viscoelastic models

❑ Behavior under harmonic loading - Complex modulus

- The linear viscoelastic behavior of the material can also be characterized experimentally by applying a sinusoidal loading of pulsation ω in shear, bending or traction-compression. The complex modulus is calculated from this type of loading.



$$\varepsilon(t) = \varepsilon_0 \sin(\omega t) \quad \Leftrightarrow \quad \sigma(t) = \sigma_0 \sin(\omega t + \delta)$$



- δ represents the phase shift between the two signals due to the viscoelastic nature of the material

Time-dependent behavior: linear viscoelastic models

$$\varepsilon(t) = \varepsilon_0 \sin \omega t \qquad \sigma(t,\omega) = \sigma_0 \sin(\omega t + \delta)$$

$$\sigma(t,\omega) = \sigma_0 \sin \omega t \cos \delta + \sigma_0 \cos \omega t \sin \delta$$
$$= \varepsilon_0 [E'(\omega) \sin \omega t + E''(\omega) \cos \omega t]$$

storage modulus ***loss modulus***

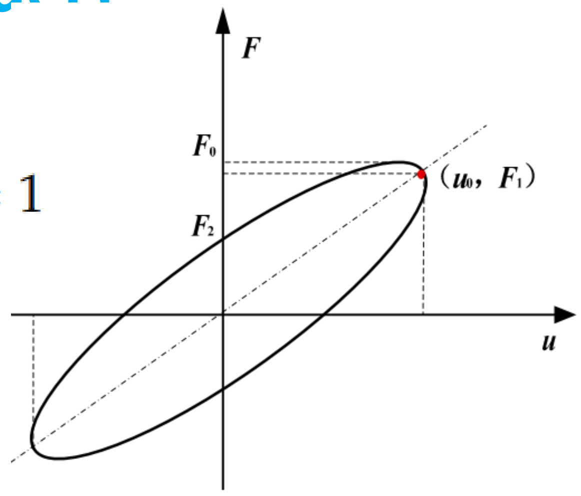
$$E'(\omega) = (\sigma_0/\varepsilon_0) \cos \delta \qquad E''(\omega) = (\sigma_0/\varepsilon_0) \sin \delta$$

$$\cos \omega t = \frac{1}{\varepsilon_0 E''(\omega)} [\sigma(t,\omega) - E'(\omega)\varepsilon(t)]$$

Elliptic equation:

$$\left(\frac{\sigma(t,\omega) - E'(\omega)\varepsilon(t)}{\varepsilon_0 E''(\omega)} \right)^2 + \left(\frac{\varepsilon(t)}{\varepsilon_0} \right)^2 = 1$$

$$\left[\frac{F - K_{d1} u}{\eta K_{d1} u_0} \right]^2 + \left[\frac{u}{u_0} \right]^2 = 1$$



Force-displacement hysteresis curve.

- **From an energy point of view:**
  - Storage modulus *E'* represents the stored energy W by the stress in phase with the strain. This energy can make its elastic deformation recover.
  - The loss modulus *E"* represents the energy lost ΔW by the transformation into heat (by molecular friction causing viscous dissipation)

$$W = \frac{1}{2} E'(\omega) \varepsilon_0^2, \qquad \Delta W = \pi E''(\omega) \varepsilon_0^2$$

Time-dependent behavior: linear viscoelastic models

- The solicitation and the response can also be expressed in the complex form by setting:

$$\sigma(t)=\sigma^{0}e^{i(\omega t+\delta)},\quad \varepsilon(t)=\varepsilon^{0}e^{i\omega t}$$

- The complex module can be deduced in form:

$$\overline{E}(\omega)=\frac{\sigma^{0}e^{i(\omega t+\delta)}}{\varepsilon^{0}e^{i\omega t}}=\frac{\sigma^{0}}{\varepsilon^{0}}e^{i\delta}=\left|\overline{E}\right|e^{i\delta}=E'+iE''$$

  - The storage modulus and loss modulus are the real an imaginary parts of the complex modulus

$$E'(\omega) = (\sigma_0/\varepsilon_0)\cos\delta$$

$$E''(\omega) = (\sigma_0/\varepsilon_0)\sin\delta$$

→ ***loss factor:*** $\tan\delta = \frac{E''}{E'}$

Time-dependent behavior: linear viscoelastic models

Complex module of rheological models

- Harmonic loading: $\sigma(t)=\sigma^0 e^{i\omega t}$ → $\varepsilon(t)=\int_{-\infty}^{t} f(t-\tau)\sigma'(\tau)\,d\tau$

Maxwell model: $f(t)=\frac{1}{E}+\frac{t}{\eta}$

$$\varepsilon(t)=\int_{-\infty}^{t} f(t-\tau)\sigma'(\tau)\,d\tau=-\frac{i\sigma^0 e^{i\omega t}(E+i\omega\eta)}{E\omega\eta}$$

$$\overline{E}=\frac{\sigma(t)}{\varepsilon(t)}=\frac{-\sigma^0 e^{i\omega t}E\omega\eta}{i\sigma^0 e^{i\omega t}(E+i\omega\eta)}=\frac{Ei\omega\eta}{E+i\omega\eta}$$

Time-dependent behavior: linear viscoelastic models

Complex module of rheological models

Maxwell model:

$$\overline{E} = \frac{\sigma(t)}{\varepsilon(t)} = \frac{-\sigma^0 e^{i\omega t} E\omega\eta}{i\sigma^0 e^{i\omega t}(E + i\omega\eta)} = \frac{Ei\omega\eta}{E + i\omega\eta}$$

$$\left|\overline{E}\right| = \frac{E\omega\eta}{\sqrt{E^2 + \omega^2\eta^2}} \qquad \varphi = \mathrm{ArcTan}\left[\frac{E}{\eta\omega}\right]$$

- In the Laplace-Carson space:

$$r_{LC} = \frac{1}{f_{LC}} = \frac{E.s.\eta}{\eta s + E}$$

- We can note: the complex model has the same form as the relaxation function in the Laplace Carson space by considering $s = i * \omega$

Time-dependent behavior: linear viscoelastic models

- Complex module of rheological models

Maxwell model:

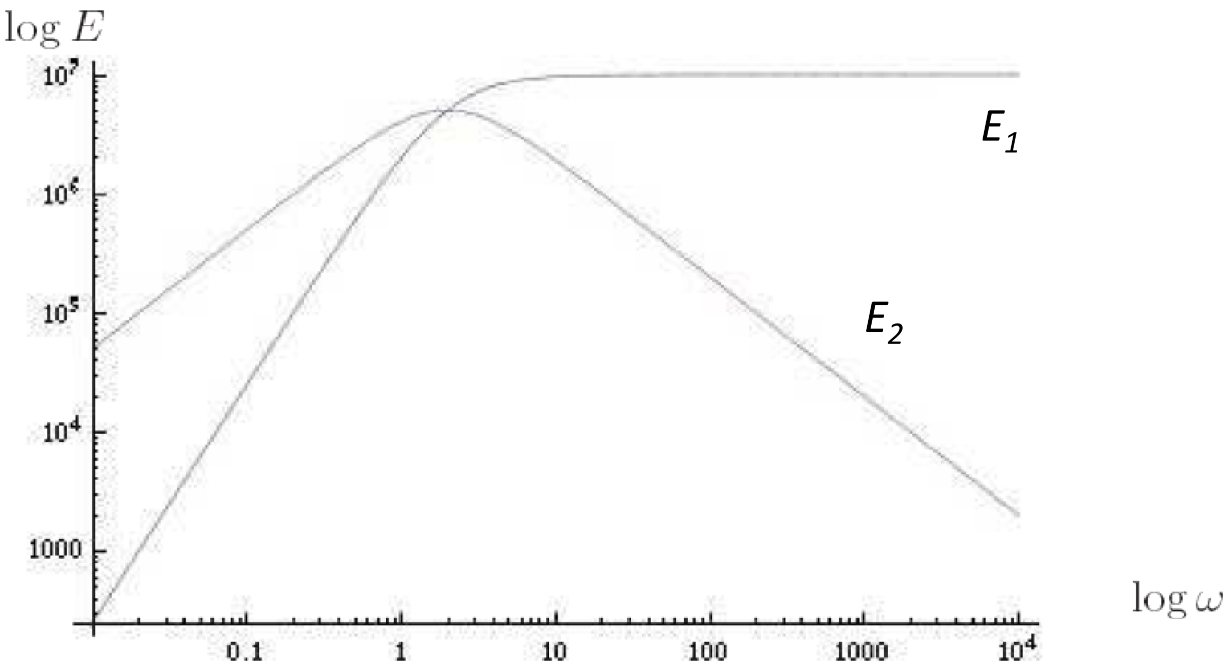
$$\overline{E} = \frac{\sigma(t)}{\varepsilon(t)} = \frac{-\sigma^0 e^{i\omega t} E\omega\eta}{i\sigma^0 e^{i\omega t}(E + i\omega\eta)} = \frac{Ei\omega\eta}{E + i\omega\eta}$$

$$E'(\omega) = E\frac{\omega^2\lambda^2}{1+\omega^2\lambda^2}, \quad \lambda = \frac{\eta}{E}$$

$$E''(\omega) = E\frac{\omega\lambda}{1+\omega^2\lambda^2},$$

$$W = \frac{E\,\eta^2\omega^2}{2\left(E^2+\eta^2\omega^2\right)}\,\varepsilon_0^2$$

$$\Delta W = \frac{E^2\eta\,\omega}{E^2+\eta^2\omega^2}\,\pi\,\varepsilon_0^2$$



Time-dependent behavior: linear viscoelastic models

- Complex module of rheological models

Kelvin model:

$$r_{LC} = E + \eta s \rightarrow \overline{E} = E + i\omega\eta$$

Kelvin-Voigt model:

$$r_{LC} = E\frac{E_K + \eta.s}{E + E_K + \eta.s} \rightarrow \overline{E} = E\frac{E_K + i\omega\eta}{E + E_K + i\omega\eta}$$

Burger model:

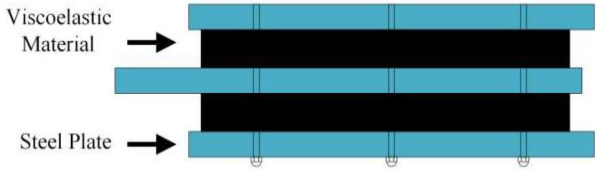
$$r_{LC} = \frac{E.s(E_K + \eta_K.s)}{E.s + (E_K + \eta_K.s)(E/\eta + s)}$$

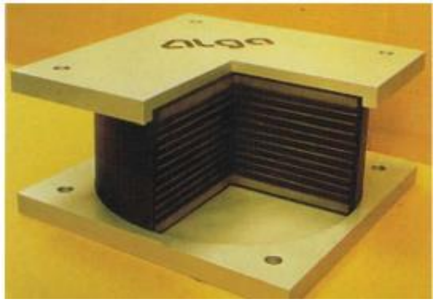
$$\rightarrow \overline{E} = \frac{E.i\omega(E_K + \eta_K.i\omega)}{E.s + (E_K + \eta_K.i\omega)(E/\eta + i\omega)}$$

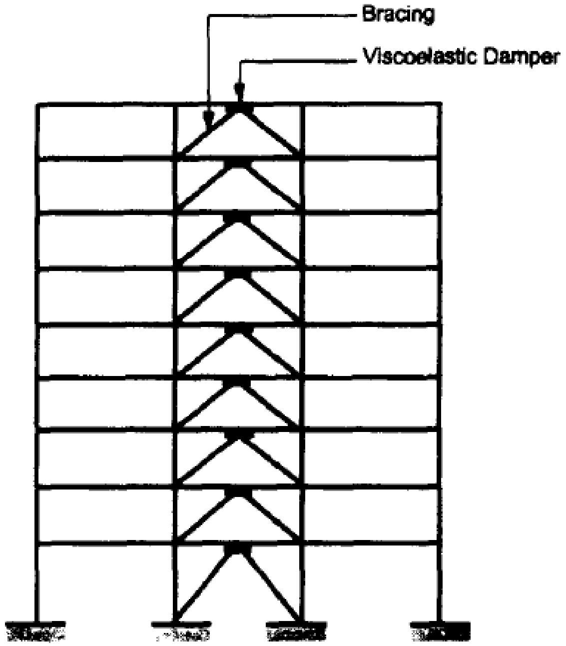
Time-dependent behavior: linear viscoelastic models

❑ Use of viscoelastic dampers in reducing wind and earthquake induced motion of building structures

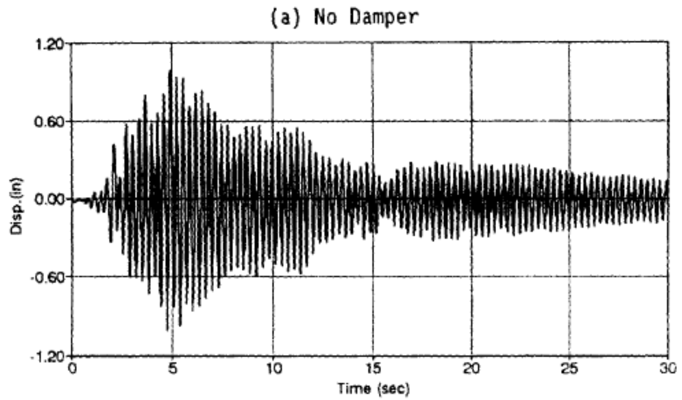
- The viscoelastic dampers can be used to reduce the building motion by converting a portion of the mechanical energy of wind or earthquake to heat.

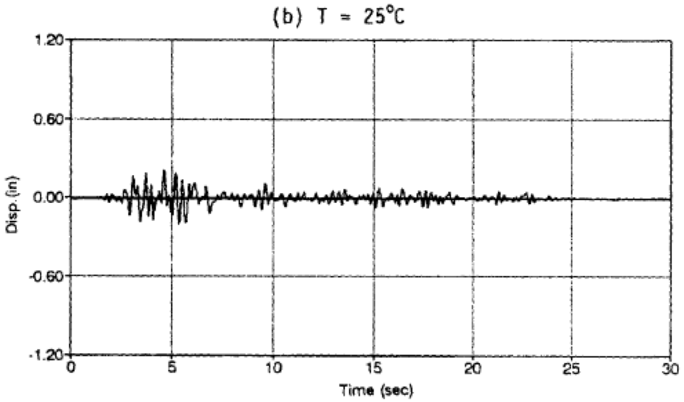






Samali and Kwok (95)

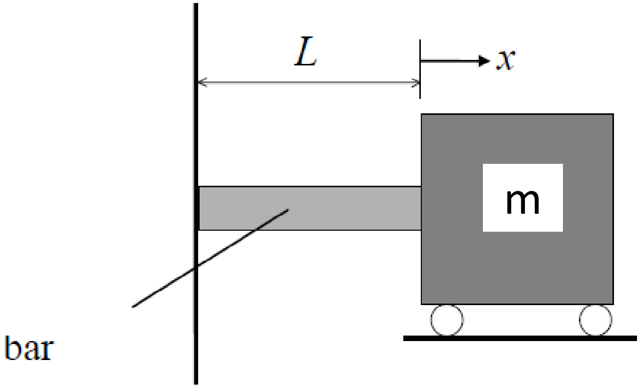




Time-dependent behavior: linear viscoelastic models

Damping of vibration

- Vibration of a simple oscillator with one degree of freedom. The mass m is connected to a wall of length L and cross section are A



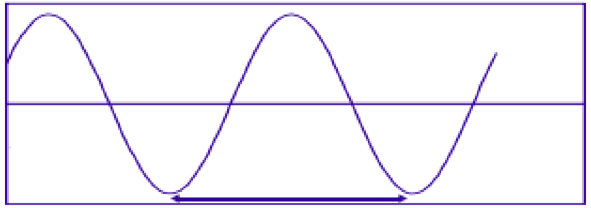
Dynamic equation: $m\ddot{x} + F = 0$

Kinematic relation: $\varepsilon = x / L$

- Supposing the elastic bar: $\sigma = E\varepsilon,$
- Assuming: $x = x_o e^{i\omega t}$

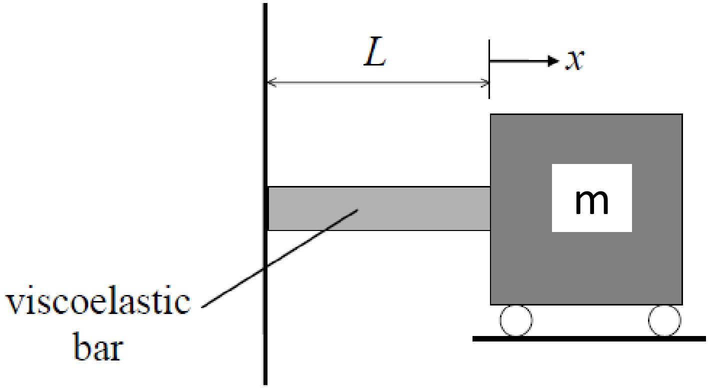
$$-\omega^2 m x_o e^{i\omega t} + A\sigma = 0 \Rightarrow -\omega^2 m x_0 e^{i\omega t} + AE\frac{x_0 e^{i\omega t}}{L} = 0 \Rightarrow -\omega^2 m + \frac{AE}{L} = 0$$

$$\Rightarrow \omega = \sqrt{\frac{AE}{mL}}, \qquad x = x_0 (A \sin \omega t + B \cos \omega t),$$



Time-dependent behavior: linear viscoelastic models

Damping of vibration



- Vibrational damping of a simple oscillator with one degree of freedom. The mass m is connected to a wall by a viscoelastic bar of length L and cross section are A

Dynamic equation: $m\ddot{x} + F = 0$

Kinematic relation: $\varepsilon = x / L$

- Supposing the Maxwell model for the viscoelastic bar:

$$\sigma = \overline{E}\varepsilon, \qquad \overline{E} = \left[\frac{1}{E} - \frac{i}{\eta\omega}\right]^{-1}$$

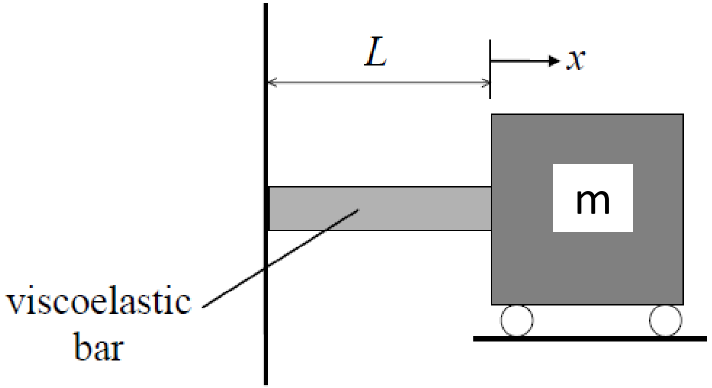
- Assuming: $x = x_o e^{i\omega t}$

$$-\omega^2 m x_o e^{i\omega t} + A\sigma = 0 \quad \Rightarrow \quad -\omega^2 m x_0 e^{iwt} + A\left(\frac{1}{E} - \frac{i}{\eta w}\right)^{-1} \frac{x_0 e^{iwt}}{L} = 0$$

Time-dependent behavior: linear viscoelastic models

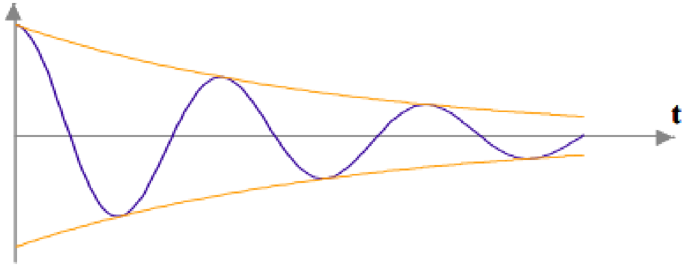
❑ Damping of vibration

$$\frac{1}{E}\omega^2 - \frac{i}{\eta}\omega = \frac{A}{Lm}, \quad \Rightarrow \quad \omega = \left\{ \frac{E}{2\eta} i \pm \sqrt{\frac{AE}{Lm} - \frac{E^2}{4\eta^2}} \right\}$$



- If *m* is small or *E* is large: $\omega = i(E/2\eta) \pm v$

$$x = x_o\left(c_1 e^{i\omega_1 t} + c_2 e^{i\omega_2 t}\right) = x_o e^{-(E/2\eta)t}\left(c_1 e^{ivt} + c_2 e^{-ivt}\right)$$
$$= x_o e^{-(E/2\eta)t}\left(A\cos(vt) + B\sin(vt)\right)$$

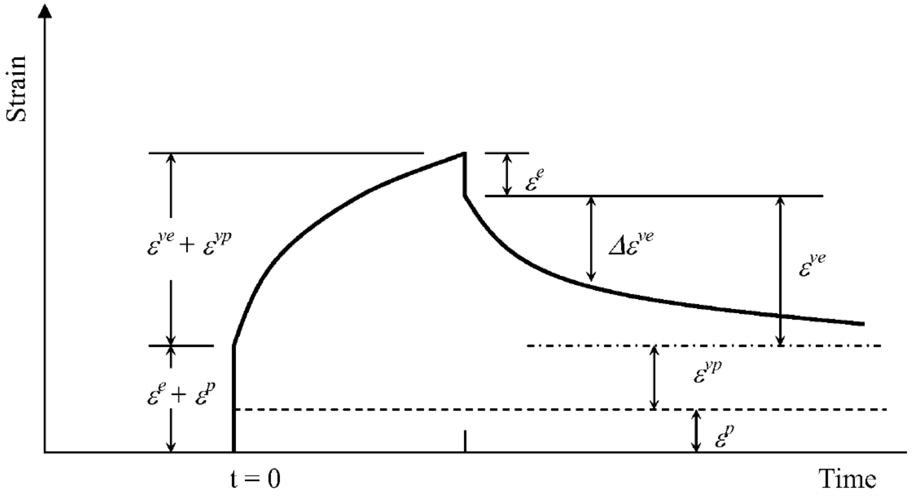


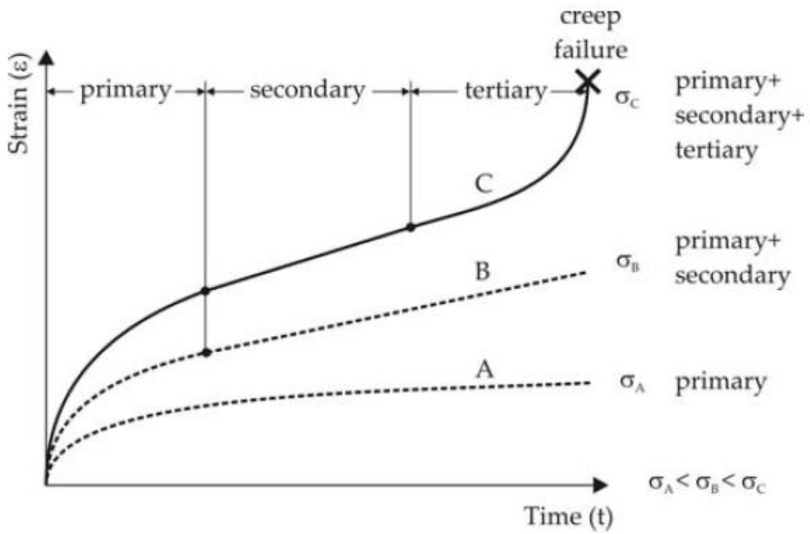
- If *m* is large or the spring compliant: $\omega = i(E/2\eta) \pm iv$

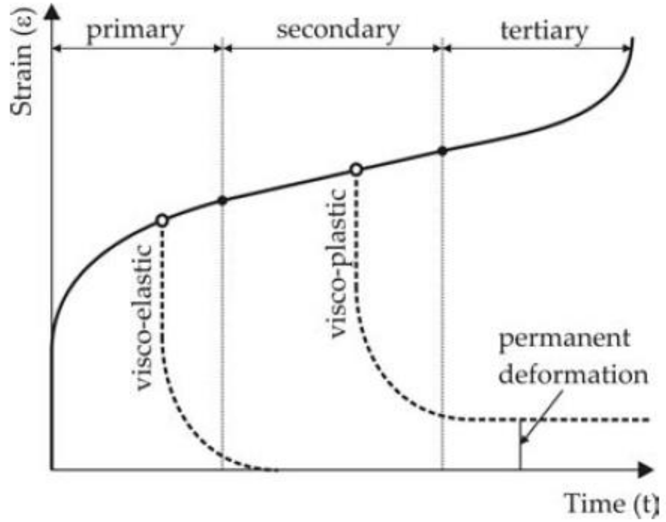
$$x = x_o\left(c_1 e^{(E/2\eta+v)t} + c_2 e^{(E/2\eta-v)t}\right)$$

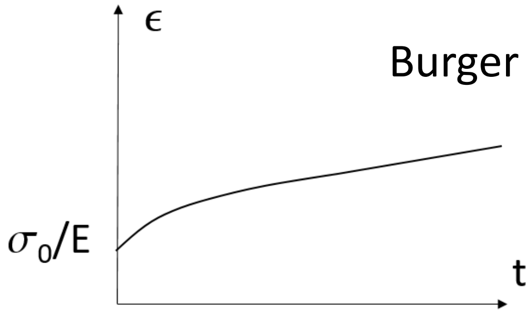
Non-linear viscoelastic and elasto-viscoplastic models

- Laboratory and in-situ observations show a very complex time-dependent behavior of the material and in many cases, linear viscoelastic models cannot reproduce these observations

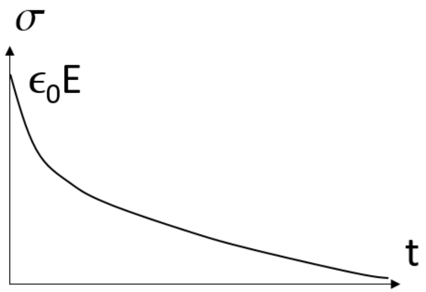






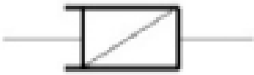


Burger model



Non-linear viscoelastic and elasto-viscoplastic models

- Non linear viscoelastic models:



$$\sigma(t) = \eta^{\beta} D^{\beta}[\varepsilon(t)], \quad 0 \leqslant \beta \leqslant 1$$

$$\begin{cases} \sigma(t) = \sigma \ \text{(constant stress)} \\ \varepsilon(t) = \dfrac{\sigma}{\eta^{\beta}} \dfrac{t^{\beta}}{\Gamma(\beta+1)}, \quad 0 \leqslant \beta \leqslant 1 \end{cases}$$

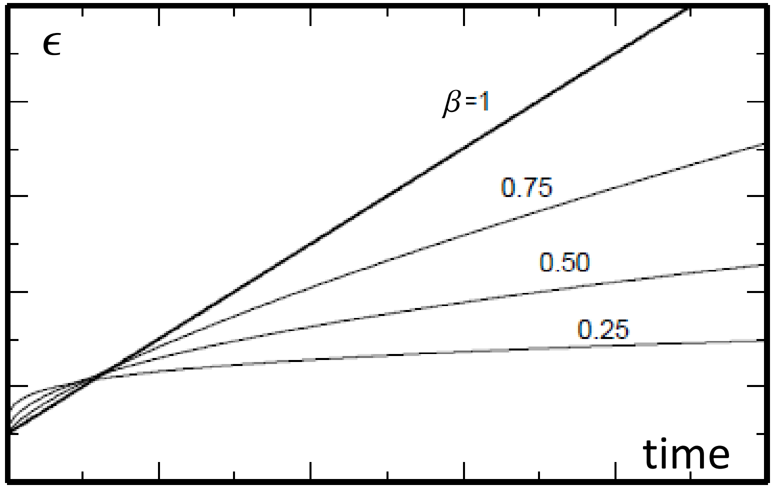
Gamma function: $\Gamma(\beta) = \int_0^{\infty} e^{-t} t^{\beta-1} dt \quad (\mathrm{Re}(\beta) > 0)$

- **Fractional Maxwell model**

$$\varepsilon(t) = \varepsilon_1 + \varepsilon_2(t)$$

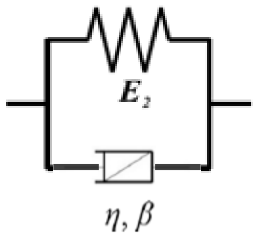
$$\varepsilon_1 = \frac{\sigma}{E}, \quad \varepsilon_2(t) = \frac{\sigma}{\eta} \frac{t^{\beta}}{\Gamma(1+\beta)}, \quad 0 \le \beta \le 1.$$

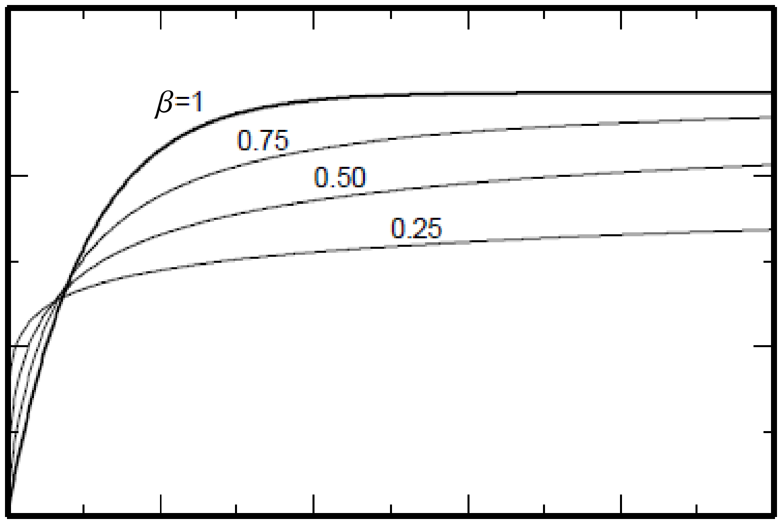
$$\varepsilon(t) = f(t).\sigma, \qquad f(t) = \frac{1}{E} + \frac{t^{\beta}}{\eta.\Gamma(1+\beta)}$$



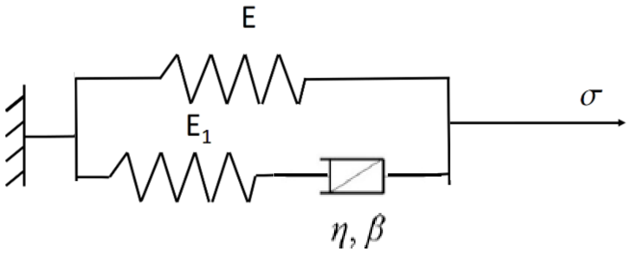
Non-linear viscoelastic and elasto-viscoplastic models

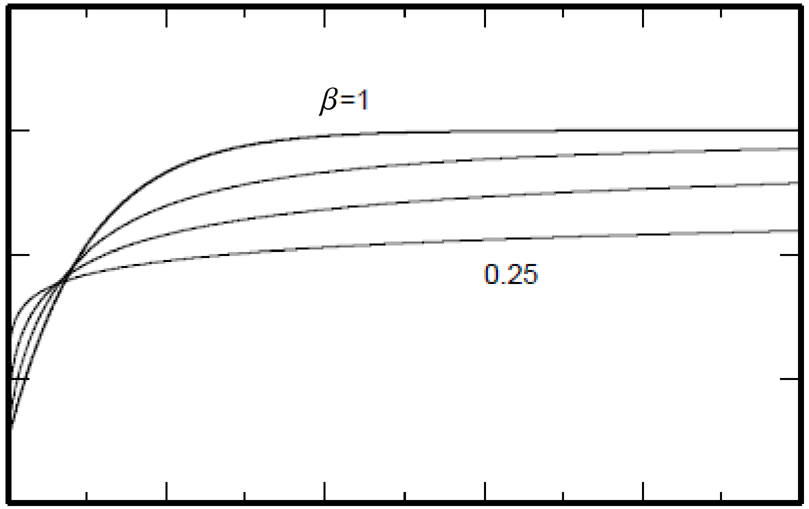
- **Fractional Kelvin model**



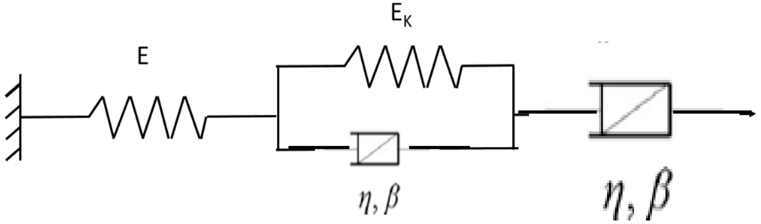


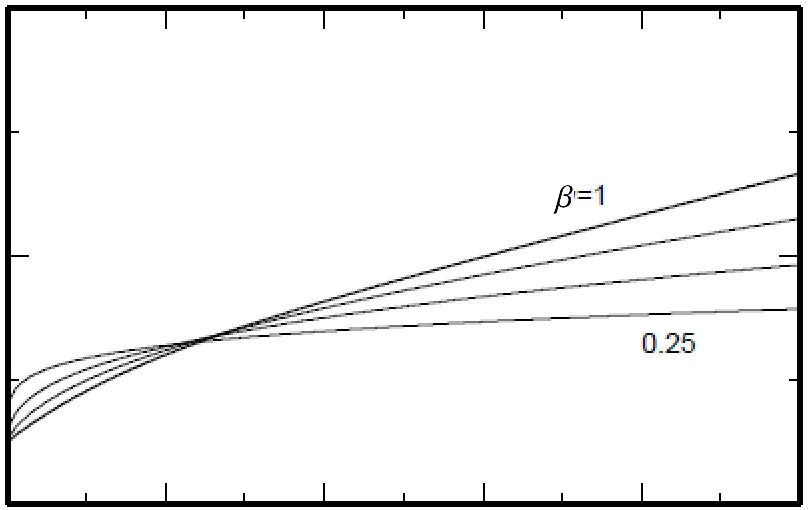
- **Fractional Kelvin-Voigt model:**





- **Fractional Burger model:**

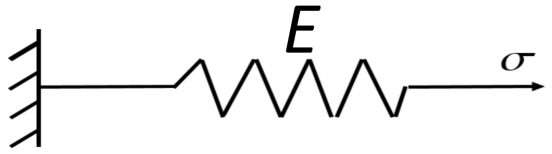




Non-linear viscoelastic and elasto-viscoplastic models

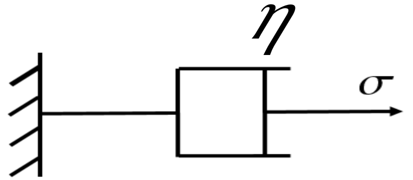
Elasto-viscoplastic models:

- The model of the spring with constant stiffness E is used to present the linear elastic behavior of the material

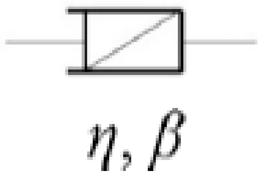


$$\sigma(t) = E.\varepsilon(t)$$

- The linear or non-linear viscous behavior is modeled by the dashpot:

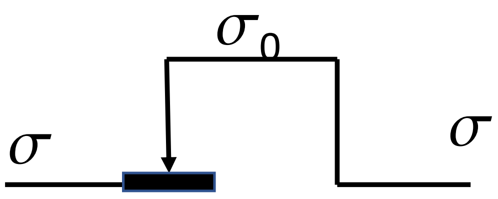


$$\sigma(t) = \eta.\varepsilon'(t)$$



$$\sigma(t) = \eta^{\beta} D^{\beta}[\varepsilon(t)], \quad 0 \leqslant \beta \leqslant 1$$

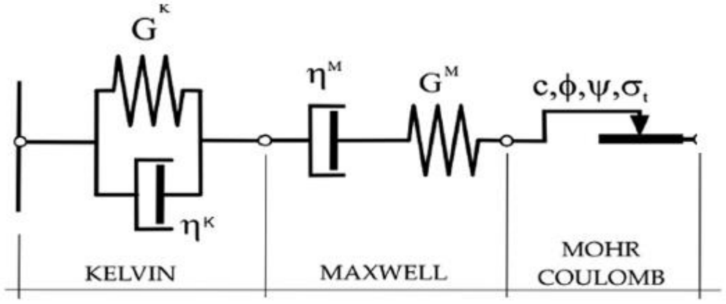
- The plastic behavior is modeled by a plastic slider:

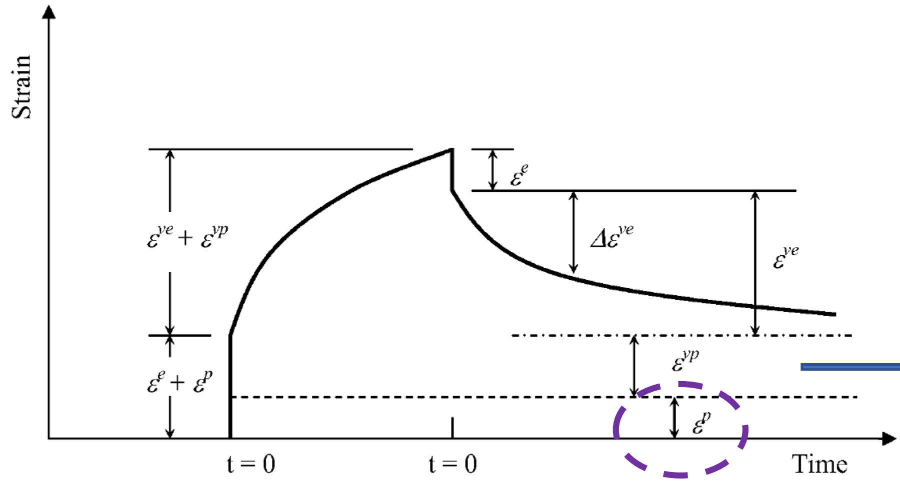


$$\sigma < \sigma_0 \Rightarrow \varepsilon = 0$$
$$\sigma = \sigma_0 \Rightarrow \varepsilon \neq 0$$

Non-linear viscoelastic and elasto-viscoplastic models

- Elasto-viscoplastic models:





=0

Permanent strain

$$\varepsilon_{ij} = \varepsilon_{ij}^{ve} + \varepsilon_{ij}^{p},$$

$$\varepsilon_{ij}^{ve} = \varepsilon_{ij}^{veK} + \varepsilon_{ij}^{veM},$$

$$s_{ij} = 2G^K e_{ij}^{veK} + 2\eta^K \dot{e}_{ij}^{veK},$$

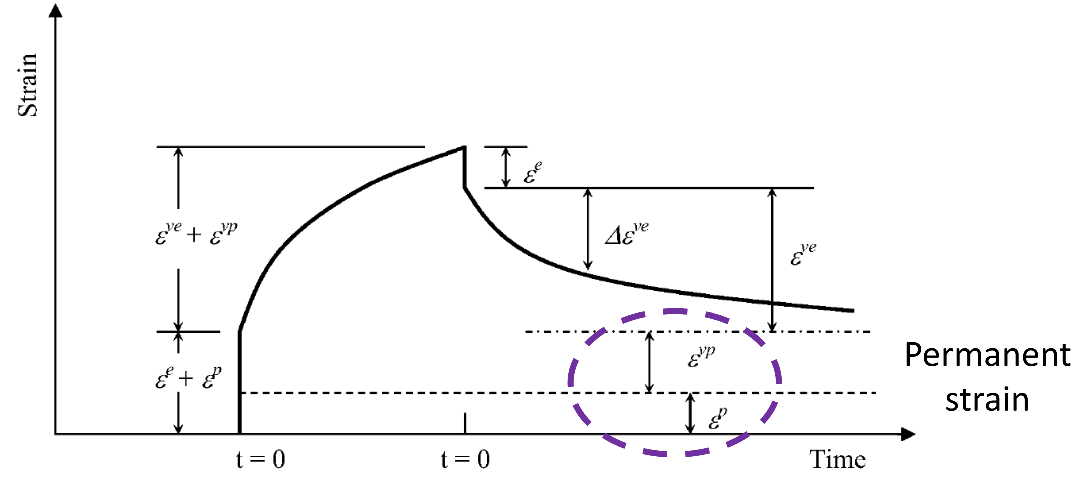
$$\dot{e}_{ij}^{veM} = \frac{\dot{s}_{ij}}{2G^M} + \frac{s_{ij}}{2\eta^M}$$

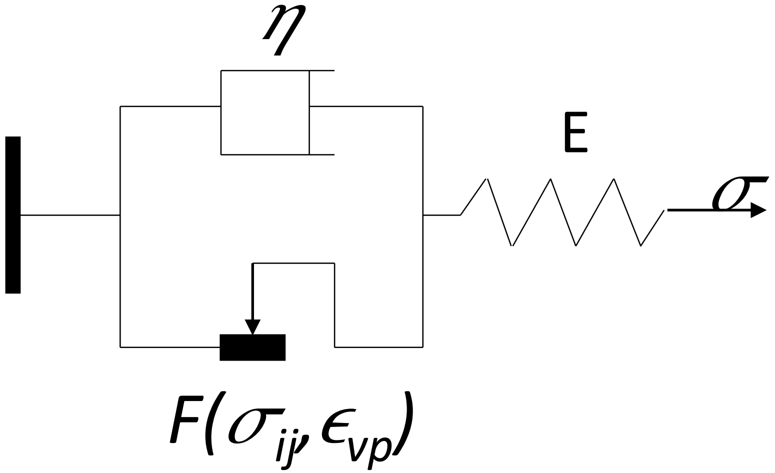
- The plastic strains follow the flow rule of plasticity:

$$\dot{\varepsilon}_{ij}^{p} = \lambda \frac{\partial g}{\partial \sigma_{ij}}$$

Non-linear viscoelastic and elasto-viscoplastic models

- Elasto-viscoplastic models:





$$\dot{\underline{\underline{\varepsilon}}}^{vp} = \frac{\partial\Omega(\underline{\underline{\sigma}}\,;\underline{\underline{\varepsilon}}^{vp})}{\partial\underline{\underline{\sigma}}} = \eta\kappa(\underline{\underline{\varepsilon}}^{vp}) < \Phi(F) > \frac{\partial F}{\partial\underline{\underline{\sigma}}}$$

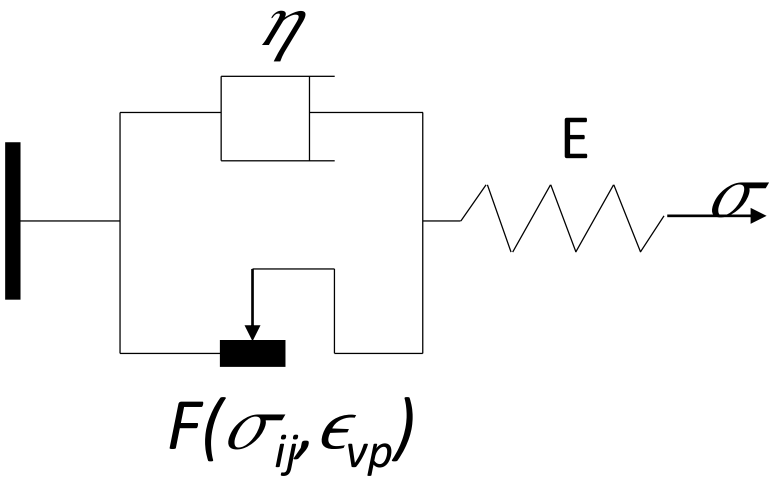
$\Omega(\underline{\underline{\sigma}}\,;\underline{\underline{\varepsilon}}^{vp})$: viscoplastic potential

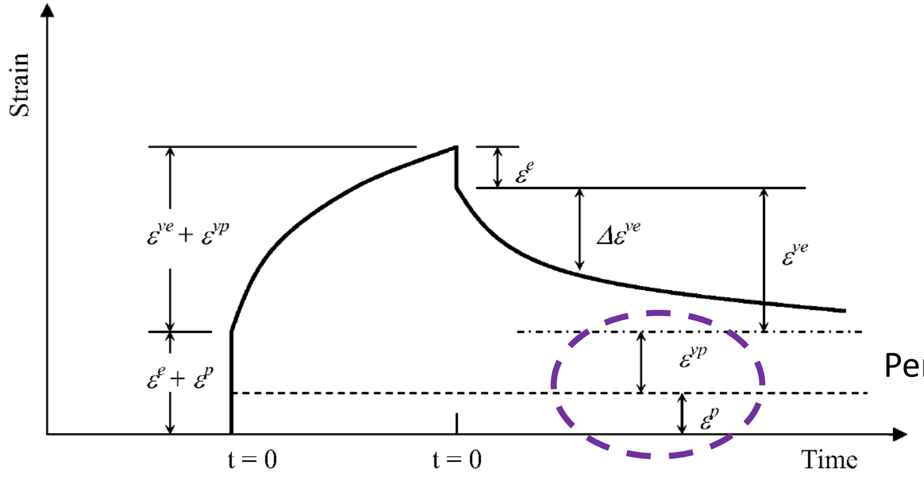
$\Phi(F)$: flow rule

F : yield surface

Non-linear viscoelastic and elasto-viscoplastic models

- Elasto-viscoplastic models:





$$\dot{\varepsilon} = \dot{\varepsilon}^{el} + \dot{\varepsilon}^{vp}$$

$$\dot{\varepsilon}^{el} = \frac{1}{E}\dot{\sigma}$$

$$\Omega = \frac{m}{m+1} k \left\langle \frac{\bar{\sigma} - \sigma_0}{k} \right\rangle^{\frac{1}{m}}$$

$$\dot{\bar{\varepsilon}} = \sqrt{\frac{2}{3}\dot{\boldsymbol{\varepsilon}}^{vp} : \dot{\boldsymbol{\varepsilon}}^{vp}} = \left\langle \frac{\bar{\sigma} - \sigma_0}{k} \right\rangle^{\frac{1}{m}} = \frac{\partial \Omega}{\partial \bar{\sigma}}$$

$$\dot{\boldsymbol{\varepsilon}}^{vp} = \frac{3}{2\bar{\sigma}} \left\langle \frac{\bar{\sigma} - \sigma_0}{k} \right\rangle^{m} \mathbf{s} = \frac{3\dot{\bar{\varepsilon}}}{2\bar{\sigma}} \mathbf{s}$$