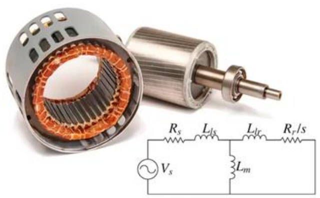


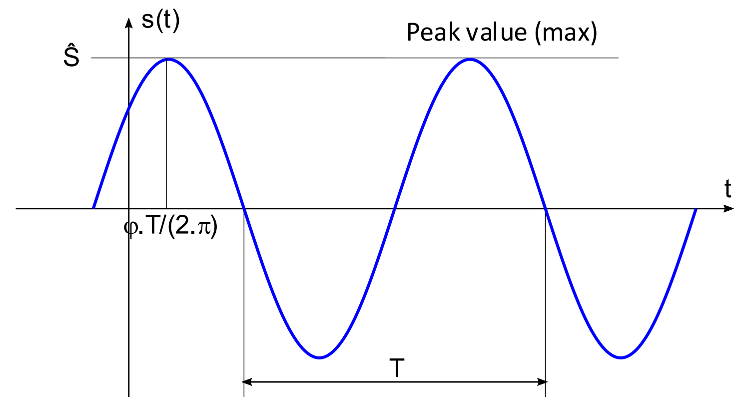
II – Power in sinusoidal regime

Sine-wave: definitions

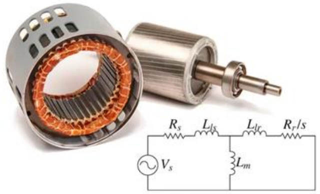
- A sinusoidal quantity s(t) (voltage, current, flux or magnetic field) is written:

$$s(t) = \hat{S}.\cos(\omega t +/- \varphi)$$

$$s(t) = S\sqrt{2}.\cos(\omega t +/- \varphi)$$



- s(t): quantity which evolves with time in sinusoidal form
- S: RMS value of the quantity,
- $\hat{S}$: maximum value reached by the quantity s(t) => $\hat{S}=S.\sqrt{2}$,
- ω : electrical pulsation in rad/s of the magnitude, $\omega= 2.\pi.f = 2.\pi/T$,
- f: signal frequency and T: signal period,
- φ : phase at origin (at t = 0).

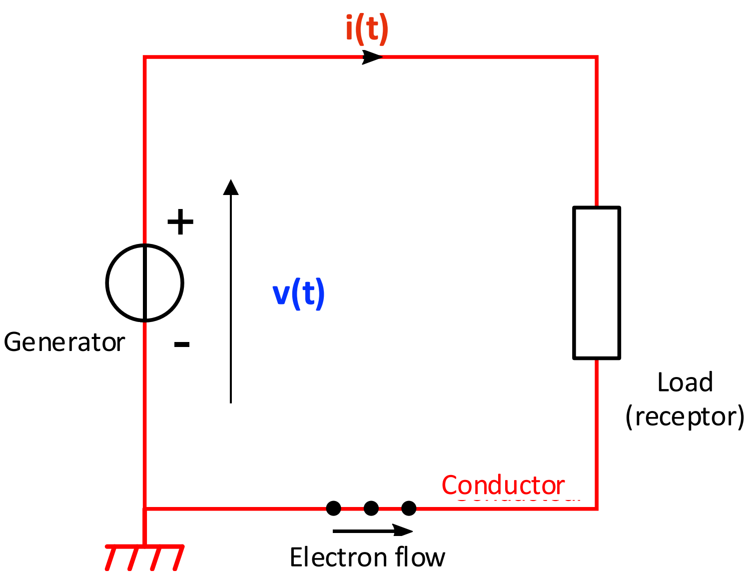


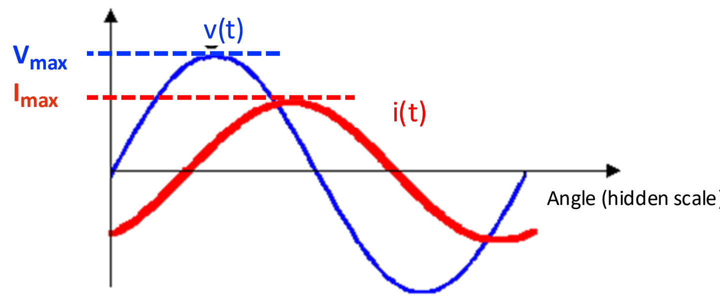
II – Power in sinusoidal regime

Single-phase

- Most of the world's electrical energy is generated, transmitted and distributed in the form of sinusoidal voltages.

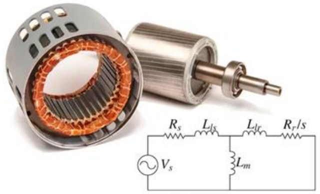
- Any periodic signal can be studied by decomposing it down into sinusoidal signals using a Fourier transform.





$$v(t) = \hat{V}.sin(\omega t)$$

$$i(t) = \hat{I}.sin(\omega t - \varphi)$$

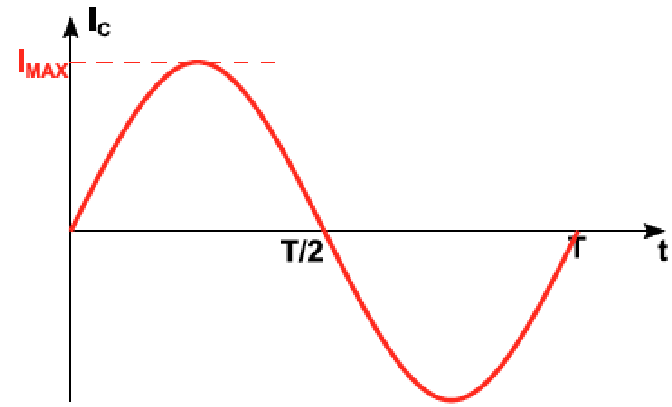


II – Power in sinusoidal regime

Single-phase – Average and RMS values

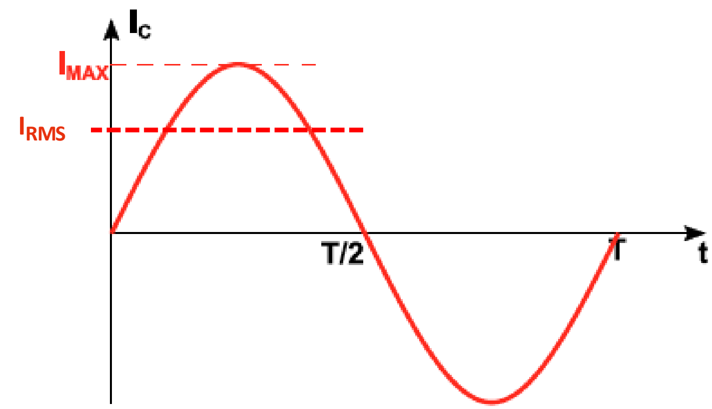
- The AVERAGE VALUE of any current (same for the voltage) is the value that a direct current carrying the same amount of electricity would have.

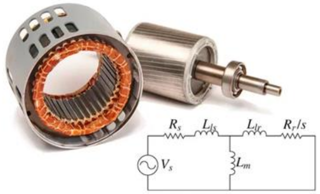
$$I_{MOY} = \frac{1}{T}\int i(t)dt \qquad I_{MOY} = 0$$



- The RMS (Root Main square) VALUE of any current is the value that a direct current carrying the same amount of energy would have.

$$I_{RMS} = \sqrt{\frac{1}{T}\int i^2(t)dt} \qquad I_{RMS} = \frac{I_{MAX}}{\sqrt{2}}$$





II – Power in sinusoidal regime

Single-phase – Average and RMS values

- Exercise: demonstrate the expression of I_{moy} and I_{RMS}

Average value

- $I_{MOY} = \frac{1}{T}\int_0^T i(t)\,dt$
- $I_{MOY} = \frac{1}{T}\int_0^T I_{MAX}.\sin(\omega\, t)\,dt$
- $I_{MOY} = \frac{1}{T}\left[I_{MAX}.\frac{-\cos(\omega t)}{\omega}\right]_0^T$
- $I_{MOY} = \frac{I_{MAX}}{\omega T}.[-\cos(\omega T) + \cos(0)]$
- $I_{MOY} = \frac{I_{MAX}}{\omega T}.(-1+1)$
- $\boldsymbol{I_{MOY} = 0}$

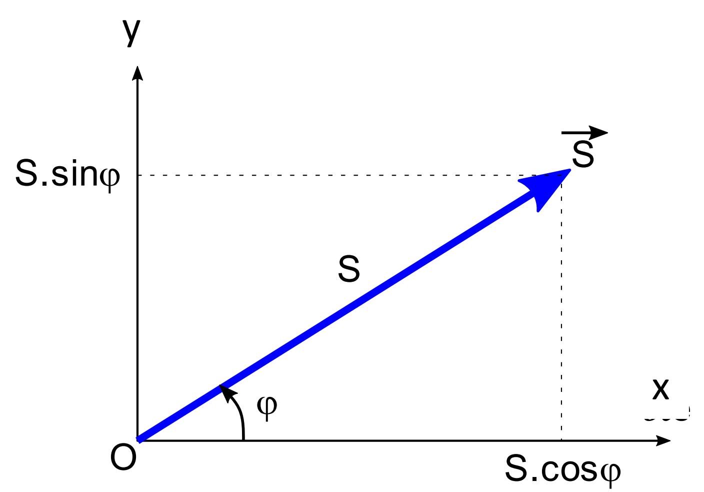
$$cos2a = 1 - 2sin^2a \;\; et \;\; sin^2a = \frac{1-cos2a}{2}$$

RMS value

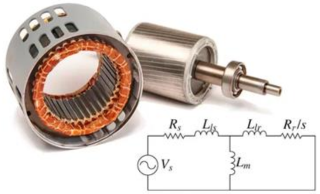
- $I_{EFF} = \sqrt{\frac{1}{T}\int_0^T i^2(t)\,dt}$
- ${I_{EFF}}^2 = \frac{1}{T}\int_0^T i^2(t)\,dt$
- ${I_{EFF}}^2 = \frac{1}{T}\int_0^T (I_{MAX}.\sin(\omega\, t))^2 dt$
- ${I_{EFF}}^2 = \frac{1}{T}\int_0^T (I_{MAX})^2.(\sin(\omega\, t))^2 dt$
- ${I_{EFF}}^2 = \frac{1}{T}\int_0^T (I_{MAX})^2 \frac{(1-\cos 2\omega t)}{2}\,dt$
- ${I_{EFF}}^2 = \frac{{I_{MAX}}^2}{2.T}\int_0^T (1-\cos 2\omega t)\,dt$
- ${I_{EFF}}^2 = \frac{{I_{MAX}}^2}{2.T}\left[\mathrm{t} - \frac{\sin(2\omega t)}{2\omega}\right]_0^T$
- ${I_{EFF}}^2 = \frac{{I_{MAX}}^2}{2}\left[\frac{\mathrm{T}-0}{T} - \frac{\sin(2\omega T)-\sin(0)}{2\omega T}\right]$
- ${I_{EFF}}^2 = \frac{{I_{MAX}}^2}{2}\left[1 - \frac{0-0}{4\pi}\right]$
- ${I_{EFF}}^2 = \frac{{I_{MAX}}^2}{2}$
- $\boldsymbol{I_{EFF} = \frac{I_{MAX}}{\sqrt{2}}}$

Single-phase – Fresnel vector representation

- Associated with s(t) is a vector $\vec{S}$ known as the Fresnel vector, of norm S (RMS value) rotating around the origin point O at an angular frequency w.



- Since all signals have the same angular frequency w, vectors in the same Fresnel diagram rotate at the same speed. Therefore, they are represented at t = 0.



II – Power in sinusoidal regime

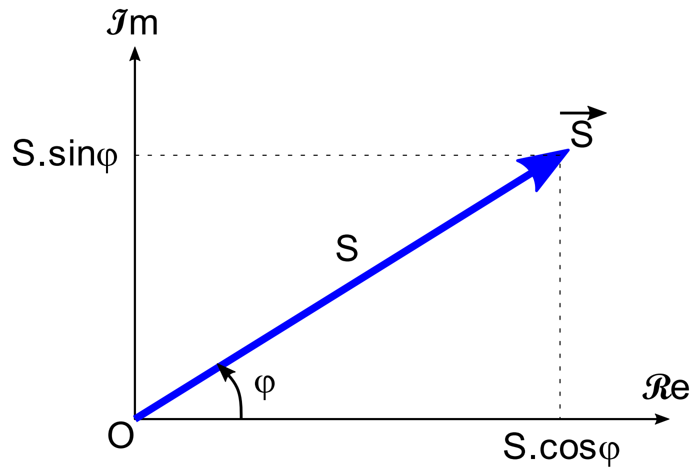
Single-phase – Complex representation

- Complex representation:

- Reminder: Euler's formula

$$e^{ix} = cos\, x + i.\, sin\, x$$

$$\boldsymbol{S.\cos\varphi + j.S.\sin\varphi = S.e^{j\varphi} = \underline{S}}$$



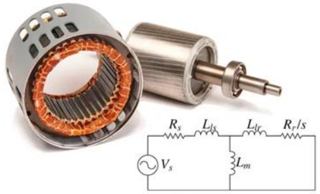
- Complex quantity

$$\begin{cases} \underline{s}(t) = S\sqrt{2}.\,e^{j(\omega t+\varphi)} \\ s(t) = \Re e\left(\underline{s}(t)\right) \end{cases}$$

- Complex amplitude

$$\underline{s} = \underline{S}\sqrt{2}.\,e^{j\omega t}$$

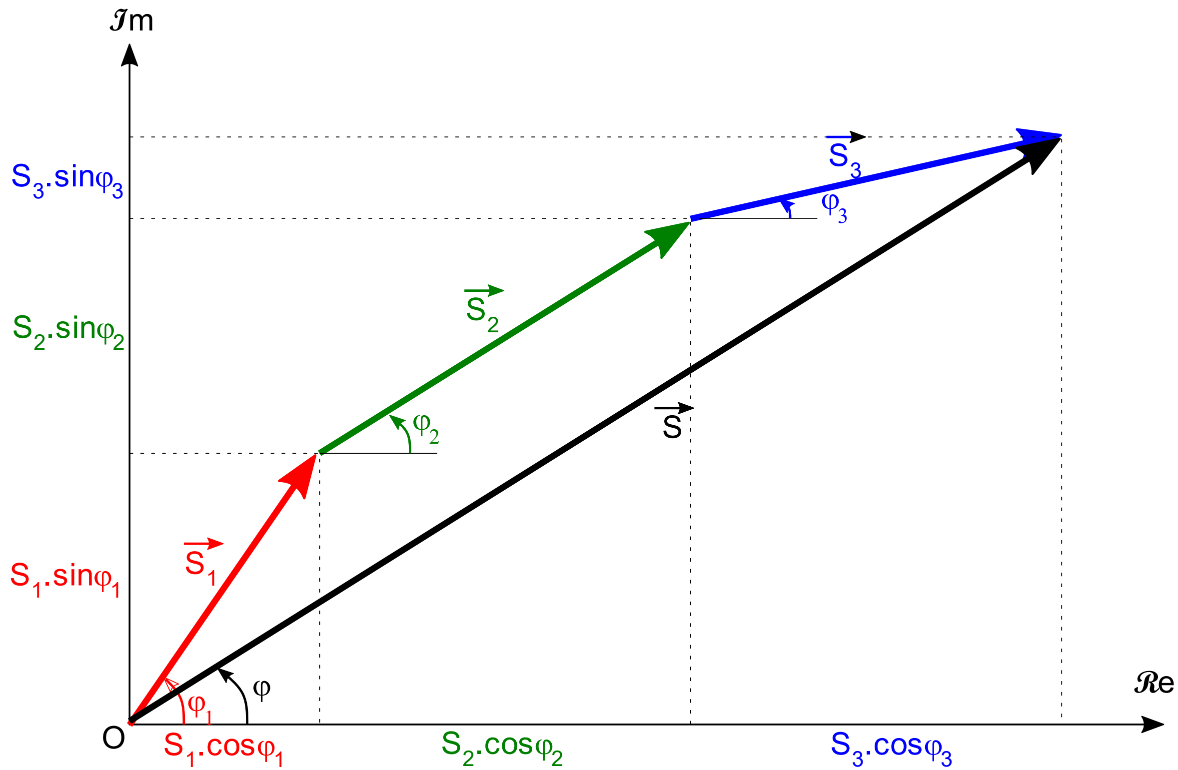
where $\underline{S} = S.\,e^{j\varphi} = [S;\varphi]$



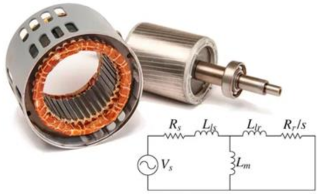
II – Power in sinusoidal regime

Single-phase – Complex representation

- Sum:



$$S = \sqrt{\left(\sum S_i.\cos\varphi_i\right)^2 + \left(\sum S_i.\sin\varphi_i\right)^2} \qquad \tan\varphi = \frac{\sum S_i.\sin\varphi_i}{\sum S_i.\cos\varphi_i}$$

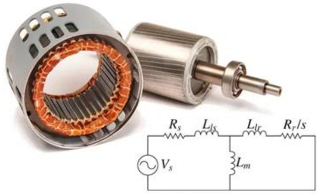


II – Power in sinusoidal regime

Single-phase – Complex representation

- Derivate: $\underline{s}(t) = \hat{S}.e^{j(\omega t+\varphi)} = \hat{S}.\cos(\omega t+\varphi) + j.\hat{S}.\sin(\omega t+\varphi)$

$$\frac{d\underline{s}(t)}{dt} = \frac{d\big(\hat{S}.\cos(\omega t+\varphi) + j.\hat{S}.\sin(\omega t+\varphi)\big)}{dt}$$



II – Power in sinusoidal regime

Single-phase – Complex representation

- Derivate: $\underline{s}(t) = \hat{S}.e^{j(\omega t+\varphi)} = \hat{S}.\cos(\omega t+\varphi) + j.\hat{S}.\sin(\omega t+\varphi)$

$$\frac{d\underline{s}(t)}{dt} = \frac{d\left(\hat{S}.\cos(\omega t+\varphi) + j.\hat{S}.\sin(\omega t+\varphi)\right)}{dt}$$

$$\frac{d\underline{s}(t)}{dt} = \hat{S}.[-\omega.\sin(\omega t+\varphi) + j.\omega.\cos(\omega t+\varphi)]$$

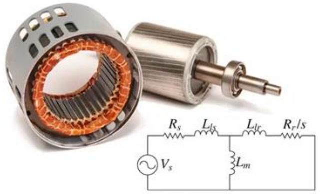
$$\frac{d\underline{s}(t)}{dt} = j.\omega.\hat{S}.\left[-\frac{\sin(\omega t+\varphi)}{j} + \cos(\omega t+\varphi)\right]$$

$$\frac{d\underline{s}(t)}{dt} = j.\omega.\hat{S}.[\cos(\omega t+\varphi) + j\sin(\omega t+\varphi)]$$

$$\boldsymbol{\frac{d\underline{s}(t)}{dt} = j.\omega.\underline{s}(t)}$$

Derivative with respect to time: **rotation of +π/2 rad** in the complex plane

$$\boldsymbol{\frac{d\underline{s}(t)}{dt} = j.\omega.\underline{s}(t) = \left[S.\omega; \varphi + \frac{\pi}{2}\right]}$$



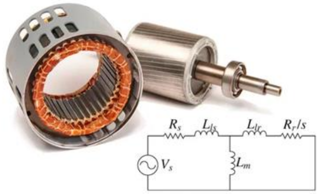
II – Power in sinusoidal regime

Single-phase – Complex representation

- Integrate:

$$\underline{s}(t) = \hat{S}.e^{j(\omega t+\varphi)} = \hat{S}.\cos(\omega t+\varphi) + j.\hat{S}.\sin(\omega t+\varphi)$$

$$\int \underline{s}(t).dt = \int \left(\hat{S}.\cos(\omega t+\varphi) + j.\hat{S}.\sin(\omega t+\varphi)\right).dt$$



II – Power in sinusoidal regime

Single-phase – Complex representation

- Integrate: $$\int \underline{s}(t).dt = \int \left(\hat{S}.\cos(\omega t+\varphi) + j.\hat{S}.\sin(\omega t+\varphi)\right).dt$$

$$\int \underline{s}(t).dt = \hat{S}.\left[\int (\cos(\omega t+\varphi)).dt + j.\int (\sin(\omega t+\varphi)).dt\right] = \hat{S}.\left[\frac{\sin(\omega t+\varphi)}{\omega} + j.\frac{-\cos(\omega t+\varphi)}{\omega}\right]$$

$$\int \underline{s}(t).dt = \frac{\hat{S}}{j\omega}.[j.\sin(\omega t+\varphi) + j.j.(-\cos(\omega t+\varphi))]$$

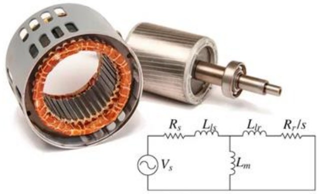
$$\int \underline{s}(t).dt = \frac{\hat{S}}{j\omega}.[j.\sin(\omega t+\varphi) + \cos(\omega t+\varphi)]$$

$$\int \underline{s}(t).dt = \frac{1}{j\omega}.\hat{S}.[\cos(\omega t+\varphi) + j.\sin(\omega t+\varphi)]$$

$$\int \underline{s}(t).dt = \frac{\underline{s}(t)}{j.\omega}$$

Time integration: **rotation of -π/2 rad** in the complex plane

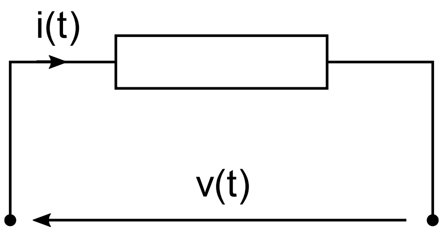
$$\boldsymbol{\int \underline{s}(t).dt = \frac{\underline{s}(t)}{j.\omega} = \left[\frac{S}{\omega};\varphi - \frac{\pi}{2}\right]}$$

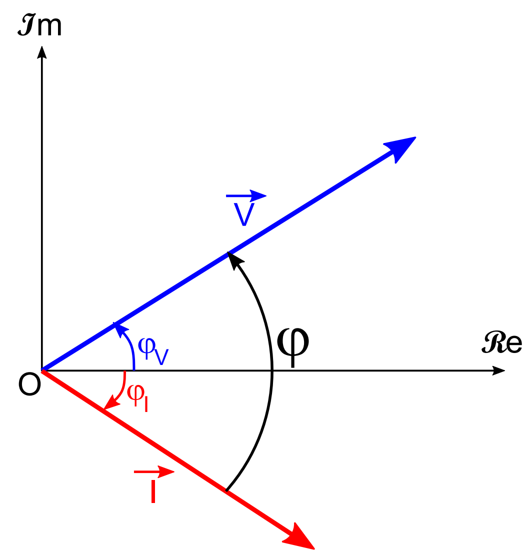


II – Power in sinusoidal regime

Single-phase – Complex impedances

- Let us consider a passive linear dipole, in receptor convention, subjected to a sinusoidal voltage v(t) and flowed by a sinusoidal current i(t):

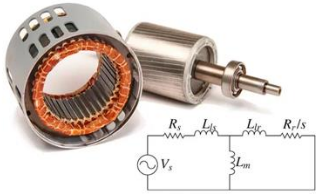




- We write:

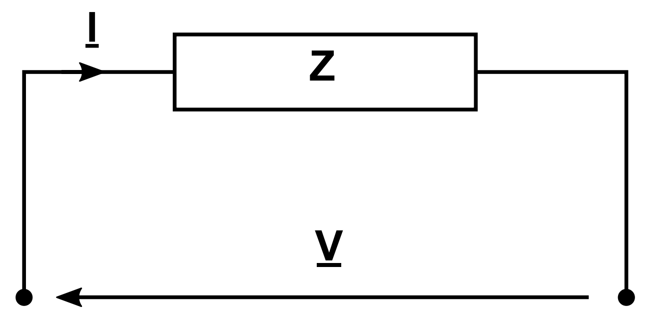
$$\begin{cases} v(t) = V\sqrt{2}.\cos(\omega t + \varphi_V) \\ i(t) = I\sqrt{2}.\cos(\omega t + \varphi_I) \end{cases}$$

$$\begin{cases} \underline{V} = V.e^{j\varphi_V} = [V;\varphi_V] \\ \underline{I} = I.e^{j\varphi_I} = [I;\varphi_I] \end{cases}$$



II – Power in sinusoidal regime

Single-phase – Complex impedances



- The complex impedance is defined as:

$$\underline{Z} = \frac{\underline{V}}{\underline{I}} = R + jX = \underline{Z} = Z.e^{j\varphi}$$

- where:

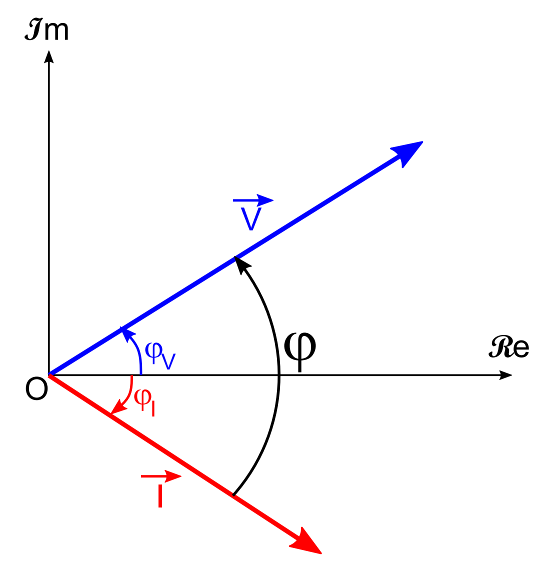
- Z : dipole impedance $Z = \|\underline{Z}\| = \frac{V}{I}$

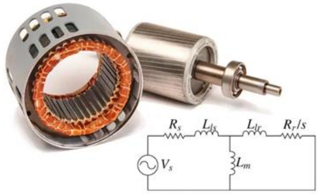
- φ : phase shift angle of voltage with respect to current:

$$\varphi = Arg\underline{Z} = \varphi_V - \varphi_I$$

- R: resistance in Ohms of the dipole, real part of the complex impedance: $R = \Re e(\underline{Z}) = Z.cos\varphi$

- X : reactance in Ohms of the dipole, imaginary part of the complex impedance $X = \Im m(\underline{Z}) = Z.sin\varphi$





II – Power in sinusoidal regime

Single-phase – Complex impedances

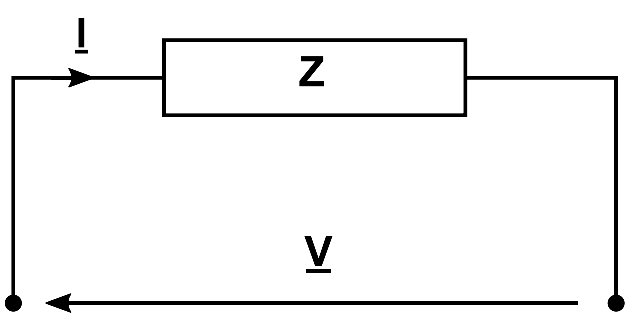
$$\underline{Z} = \frac{\underline{V}}{\underline{I}} = R + jX = \underline{Z} = Z.e^{j\varphi}$$

- where:

- Z : dipole impedance $Z = \left\|\underline{Z}\right\| = \frac{V}{I}$

- φ : phase shift angle of voltage with respect to current:

$$\varphi = Arg\underline{Z} = \varphi_V - \varphi_I$$



- If $\varphi > 0$, the voltage leads the current, impedance or load is said to be inductive.

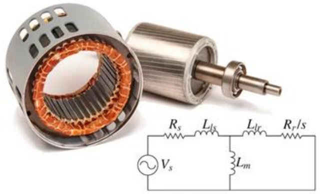
(Motors, Transformers, Electromagnets)

- If $\varphi = 0$, the voltage in in phase with the current, impedance or load is said to be resistive.

(Resistors, Furnaces, Regulated baths)

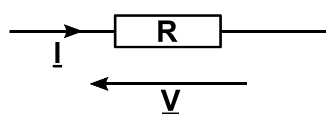
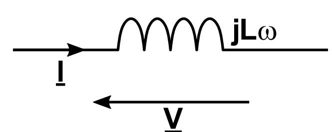
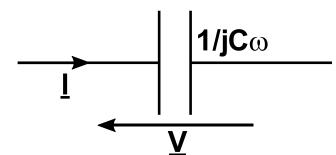
- If $\varphi < 0$, the voltage lags the current, impedance or load is said to be capacitive.

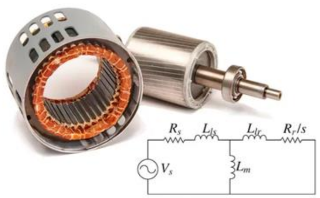
(Capacitors)



II – Power in sinusoidal regime

Single-phase – linear dipoles

	Resistor	Ideal coil	Ideal capacitor
Definition	An electrical resistor is an electrical dipole that opposes the flow of current	A "coil" is an electrical dipole that opposes the variation of electric current and can store energy in electromagnetic form.	A "capacitor" is an electrical dipole that resists variation in the voltage to which it is subjected, and can store energy in electrostatic form.
Characteristics	R : resistance in Ω (Ohm)	L : Coil inductance in H (Henry)	C : Capacitance in F (Farad)
AC regime	$V_R(t) = R.i(t)$	$V_L(t) = L.\frac{di(t)}{dt}$	$V_C(t) = \frac{1}{C}.\int i(t).dt$
Symbolic representation in receptor convention			
Sinusoidal regime	$\underline{V_R} = R.\underline{I}$	$\underline{V_L} = jL\omega.\underline{I}$	$\underline{V_C} = \frac{1}{jC\omega}.\underline{I}$
Impedance	$\underline{Z} = \mathrm{R} = [R;0]$	$\underline{Z} = \mathrm{j}L\omega = \left[L\omega;\frac{\pi}{2}\right]$	$\underline{Z} = \frac{1}{jC\omega} = \left[\frac{1}{jC\omega};-\frac{\pi}{2}\right]$



Single-phase – Fresnel (vector) representation

$\underline{V_R} = [R.I; 0]$

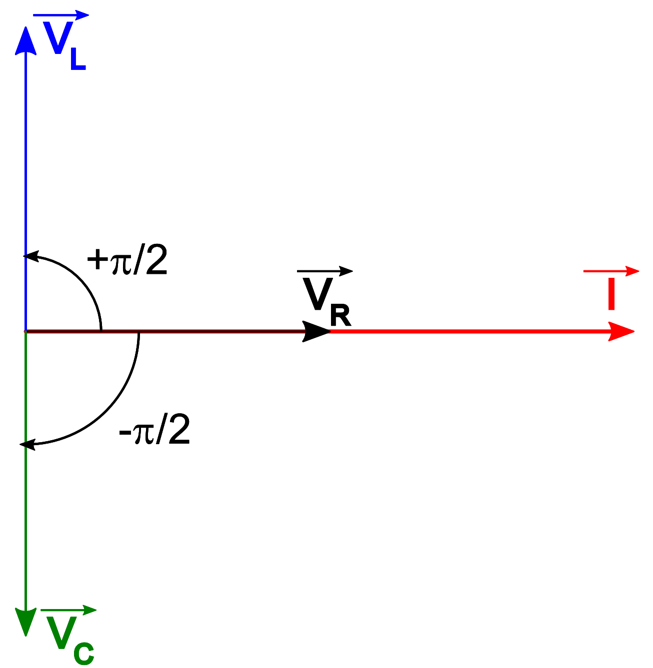
V_R and I are in phase

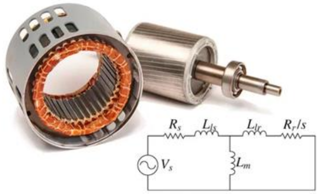
$\underline{V_L} = [L\omega.I; \frac{\pi}{2}]$

V_L leads I in quadrature

$\underline{V_C} = [\frac{I}{C\omega}; -\frac{\pi}{2}]$

V_C lags I in quadrature





Single-phase – Expressions of powers

- Instantaneous power (in W): $p(t) = v(t) \times i(t)$

=> exchange of energy, heat or work, between the network and the dipole at each instant

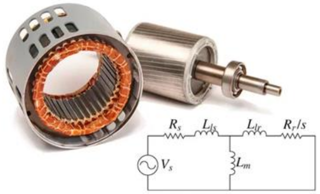
- Active power or average power (in W) – General expression:

$$P = < p(t) > = \frac{1}{T}\int_0^T p(t).dt = \frac{1}{T}\int_0^T v(t).i(t).dt$$

=> Balance of energy exchanged between the network and the dipole over a period T

=> If $P > 0$, the dipole consumes energy, it operates a receptor

=> If $P < 0$, the dipole supplies energy, it operates a generator



II – Power in sinusoidal regime

Single-phase – Expressions of powers

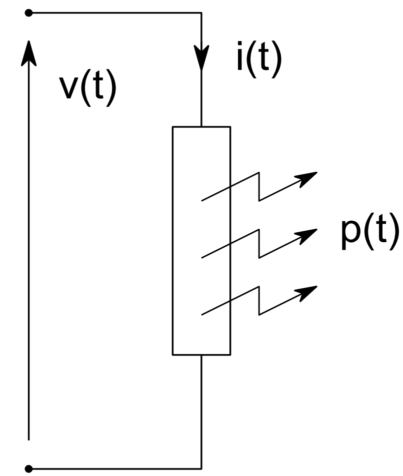
- *Linear receptor in sinusoidal regime*

- Through which a current i(t) flows

$$i(t) = I\sqrt{2} \times \sin(\omega t)$$

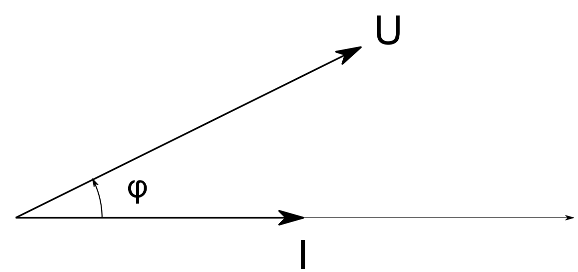
- Subjected to a voltage v(t)

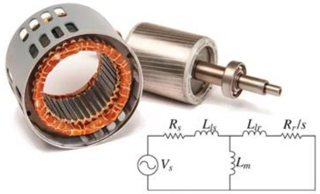
$$v(t) = V\sqrt{2} \times \sin(\omega t + \varphi)$$



φ positive, voltage ahead of current, inductive receiver
φ zero, voltage in phase with current, resistive receiver
φ negative, voltage lagging current, capacitive receiver

- Instantaneous power $p(t) = v(t) \times i(t)$

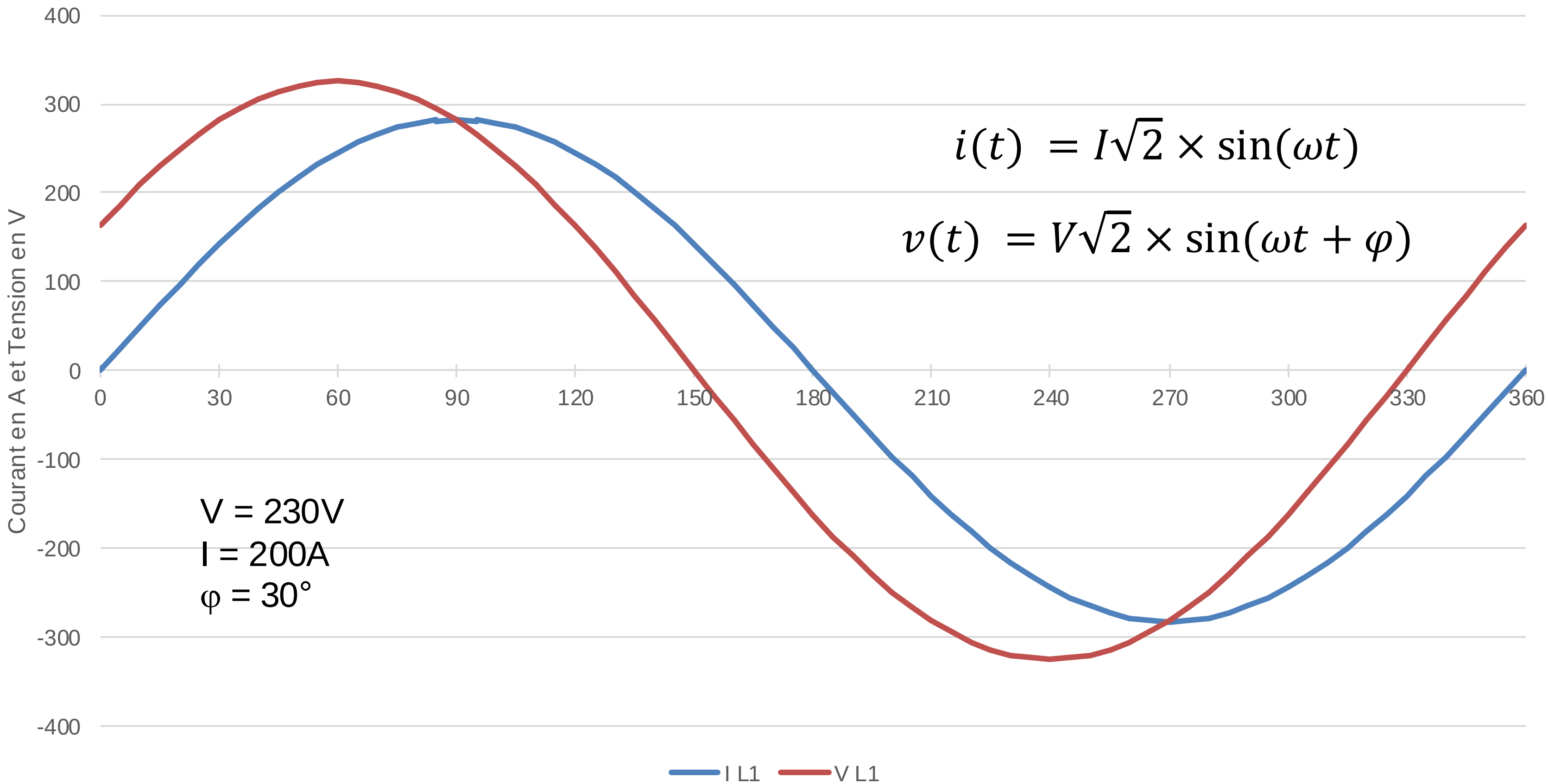


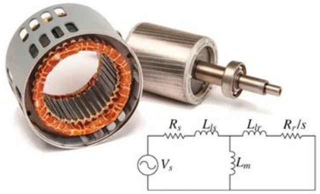


II – Power in sinusoidal regime

Single-phase – Expressions of powers

- Linear receptor in sinusoidal regime: current and voltages





II – Power in sinusoidal regime

Single-phase – Expressions of powers

- *Linear receptor in sinusoidal regime*

- Instantaneous power:

$$p(t) = v(t) \times i(t)$$

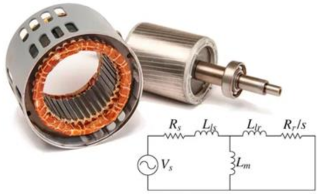
$$p(t) = V\sqrt{2} \times \sin(\omega t + \varphi) \times I\sqrt{2} \times \sin(\omega t)$$

$$p(t) = 2 \times V \times I \times \sin(\omega t + \varphi) \times \sin(\omega t)$$

$$\sin a \times \sin b = \frac{1}{2}(\cos(a-b) + \cos(a+b))$$

$$p(t) = 2 \times V \times I \times \frac{1}{2}(\cos(\omega t + \varphi - \omega t) + \cos(\omega t + \varphi + \omega t))$$

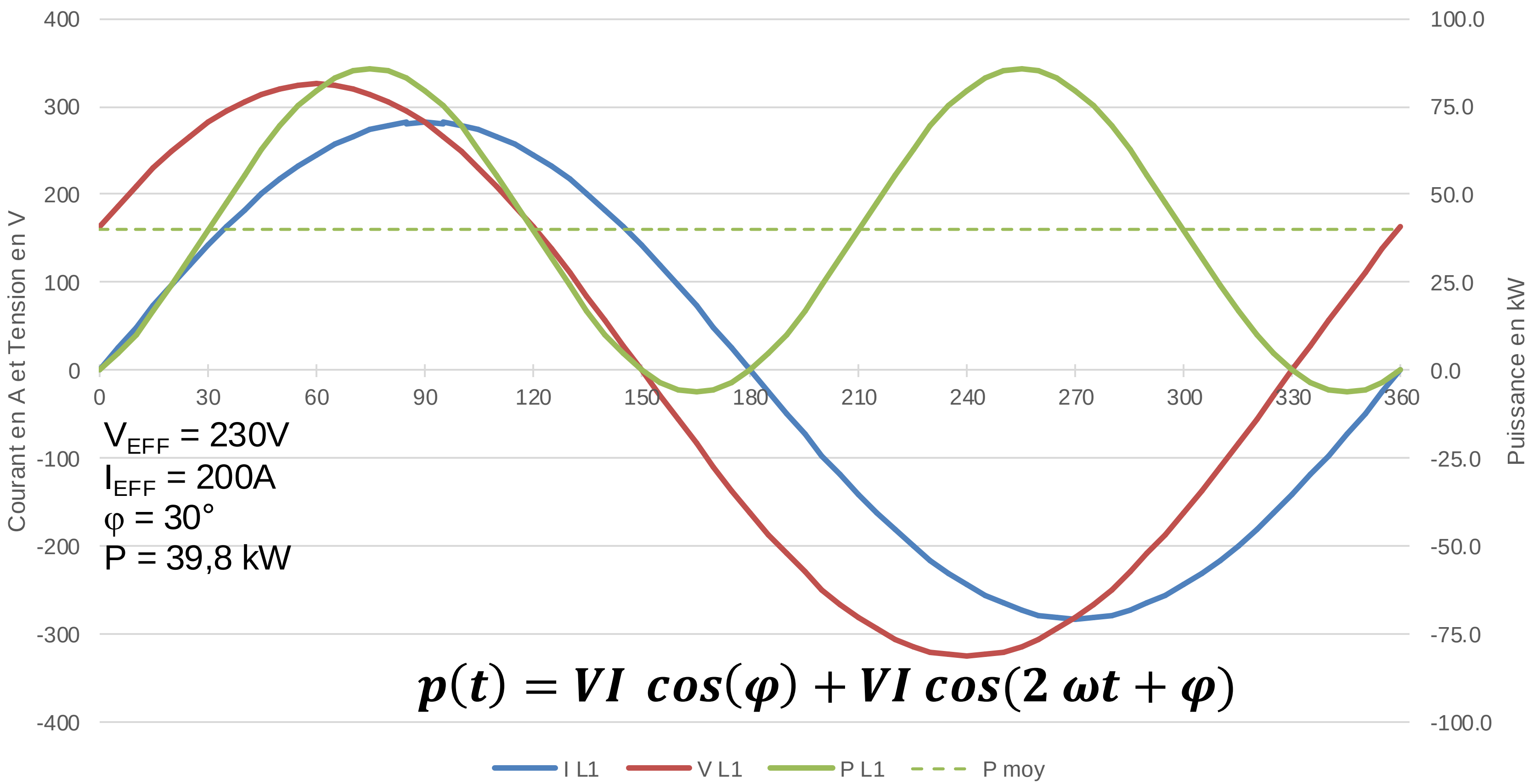
$$\boldsymbol{p(t) = V\, I\ \cos(\varphi) + V\, I \cos(2\,\omega t + \varphi)}$$



II – Power in sinusoidal regime

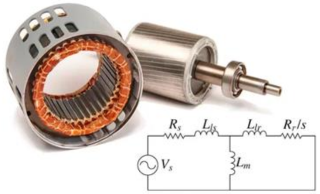
Single-phase – Expressions of powers

- Linear receptor in sinusoidal regime: current, voltage and power



$V_{EFF} = 230V$
$I_{EFF} = 200A$
$\varphi = 30°$
$P = 39{,}8$ kW

$$p(t) = VI\ \cos(\varphi) + VI\cos(2\ \omega t + \varphi)$$



II – Power in sinusoidal regime

Single-phase – Expressions of powers

- Instantaneous power (in W): $p(t) = v(t) \times i(t)$

=> exchange of energy, heat or work, between the network and the dipole at each instant

- Active power or average power (in W): $P = < p(t) > = V\ I\ \cos(\varphi)$

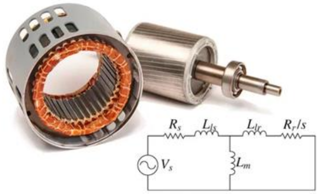
=> Balance of energy exchanged between the network and the dipole over a period T

- Apparent power (in VA): $S = V.I$

=> Design value (voltage for insulation, current for conductor cross-section)

- Power factor (general expression):

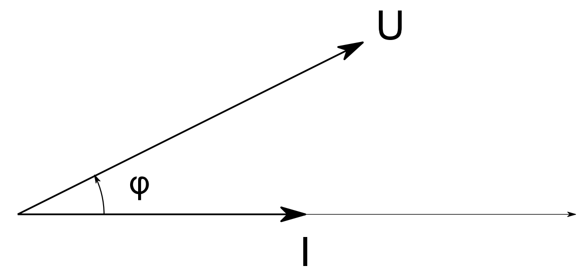
$$FP = \frac{P}{S} \qquad FP = \cos(\varphi)$$ => Sinusoidal regime only



II – Power in sinusoidal regime

Single-phase – Expressions of powers

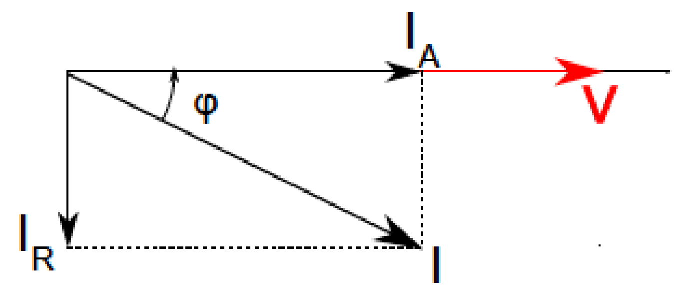
- Reactive power:



- Projection of I on the axes: $\vec{I} = \overrightarrow{I_A} + \overrightarrow{I_R}$

=> Active current: $I_A = I.cos(\varphi)$

=> Reactive current: $I_R = I.sin(\varphi)$

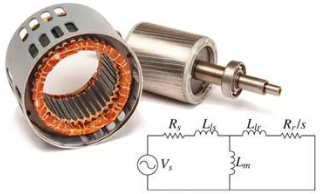


- Reminder of the active power (W):

$$P = V\ .I.cos(\varphi) = V.I_A$$

- By analogy, the reactive power (given in VAR) can be written:

$$\boldsymbol{Q = V\ .I.sin(\varphi) = V.I_R}$$



II – Power in sinusoidal regime

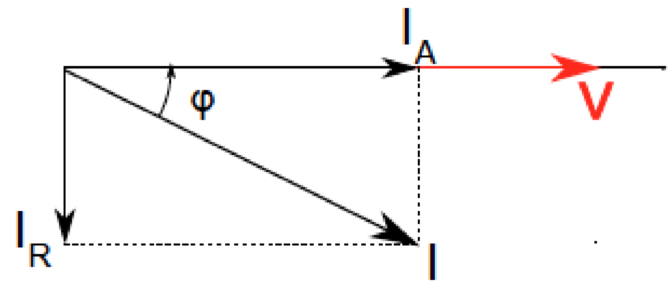
Single-phase – Expressions of powers

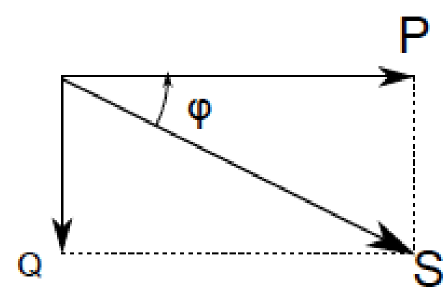
- Relation between powers S, P and Q:

$$P = V\,.I.\,cos(\varphi)$$

$$Q = V\,.I.\,sin(\varphi)\ = P.\,tan(\varphi)$$

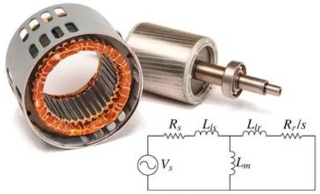
$$S^2 = P^2 + Q^2$$





- Note that: $S \neq P + Q$

- But: $\underline{S} = \underline{V}\ \underline{I}^* = P + jQ$



Single-phase – Expressions of powers

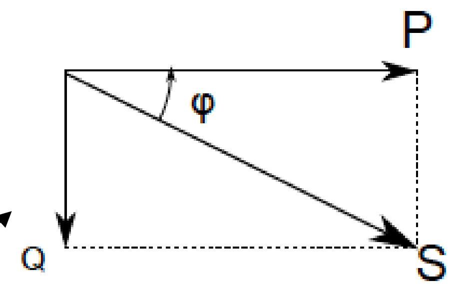
- The resistive load (e.g. a furnace)

=> $\varphi = 0$, voltage in phase with current.

=> $\cos(\varphi) = 1$, $\sin(\varphi) = 0$

=> P = V . I and Q = 0

A resistive dipole consumes no reactive power



- The inductive load (e.g. a motor)

=> The voltage leads the current, $\varphi > 0$

=> $\sin(\varphi) > 0$, **Q > 0**

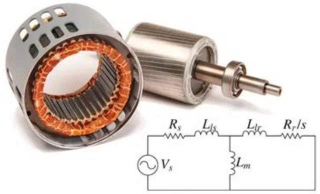
An inductive dipole consumes reactive power

- The capacitive load (e.g. a capacitor)

=> The voltage lags the current, $\varphi < 0$

=> $\sin(\varphi) < 0$, **Q < 0**

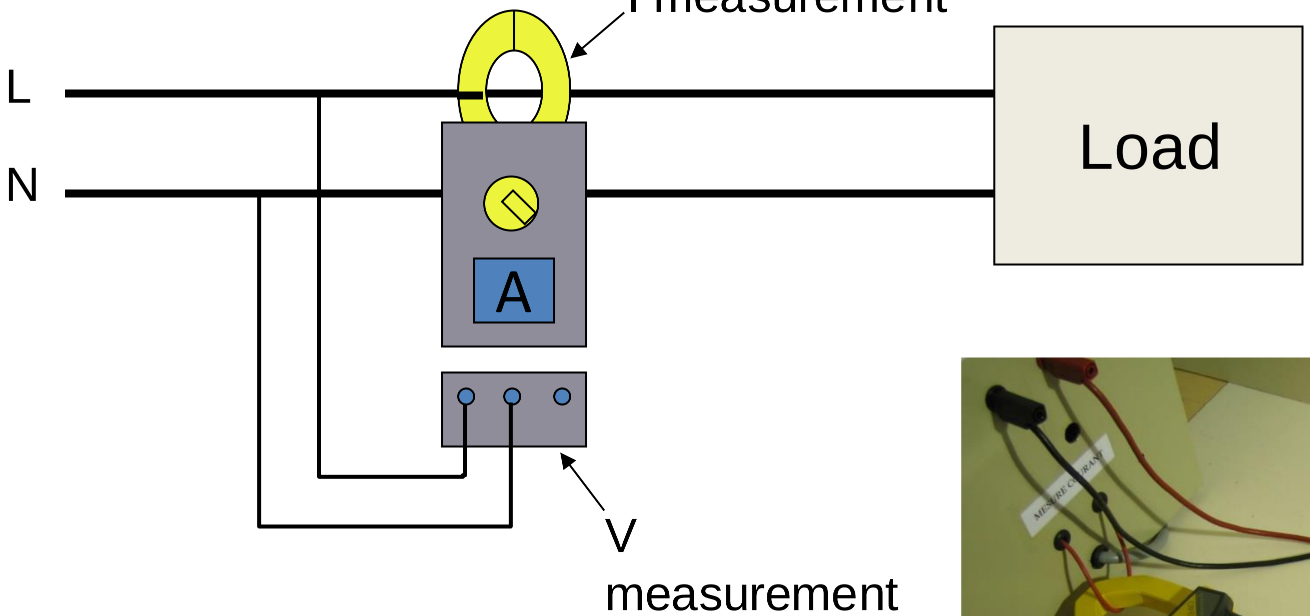
An inductive dipole supplies reactive power

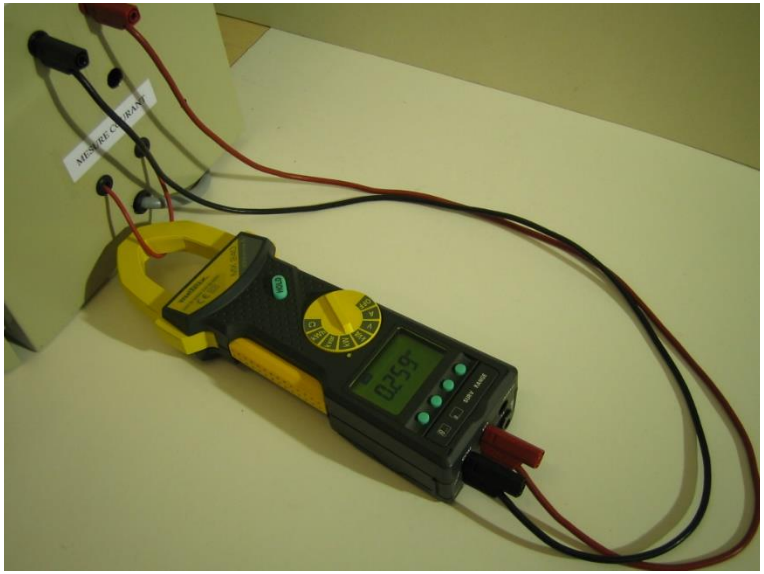


II – Power in sinusoidal regime

Single-phase

- Measurement of powers





Single-phase

- Measurement of powers