FACULTÉ DE DROIT
ÉCONOMIE & GESTION
UNIVERSITÉ D'ORLÉANS

	Licence 2 EG - Gestion (voie euro) - semestre 1, 2021-2022 *MATHEMATICS* **Midterm Exam 1** (1h30) **Documents and calculators are not allowed**
SURNAME :	Forename :

All your calculations must be sufficiently justified. It might not be necessary to do all the exercises to get 20/20

EXERCISE 1: Consider the function f defined by f: $x \mapsto \dfrac{1}{16-x^2}$ on $I=\mathbb{R}\backslash\{-4,4\}$.

1) Using the Taylor-Young theorem, check that the third-order Taylor expansion of $x \mapsto \dfrac{1}{4+x}$ at x=0 is:

$$\frac{1}{4+x} = \frac{1}{4} - \frac{x}{16} + \frac{x^2}{64} - \frac{x^3}{256} + o_{x\to 0}(x^3)$$

Let $a(x)=\frac{1}{4+x}$, then $a(0)=\frac{1}{4}$, $a'(0)=-\frac{1}{16}$, $a''(x)=2(4+x)^{-3} \Rightarrow a''(0)=\frac{2}{4^3}=\frac{1}{32}$

$a'''(x)=-6(4+x)^{-4} \Rightarrow a'''(0)=-\frac{6}{4^4}=-\frac{3}{128}$. So By Taylor-Young theorem:

$$\frac{1}{4+x}=\frac{1}{4}-\frac{1}{16}x+\frac{1}{32}\frac{x^2}{2!}-\frac{3}{128}\frac{x^3}{3!}+\underset{x\to 0}{o}(x^3)$$

$$=\frac{1}{4}-\frac{1}{16}x+\frac{x^2}{64}-\frac{x^3}{256}+\underset{x\to 0}{o}(x^3)$$

2) What is the third-order Taylor expansion of $x \mapsto \dfrac{1}{4-x}$ at x=0? Deduce from it the third-order T.E of f at x=0.

By 1), $\frac{1}{4-x}=\frac{1}{4}-\frac{1}{16}(-x)+\frac{(-x)^2}{64}-\frac{(-x)^3}{256}+\underset{x\to 0}{o}((-x)^3)=\frac{1}{4}+\frac{1}{16}x+\frac{x^2}{64}+\frac{x^3}{256}+\underset{x\to 0}{o}(x^3)$

then $f(x)=\frac{1}{16-x^2}=\frac{1}{8}\left(\frac{1}{4-x}+\frac{1}{4+x}\right)=\frac{1}{8}\left[\frac{1}{2}+\frac{x^2}{32}+\underset{x\to 0}{o}(x^3)\right]=\frac{1}{16}+\frac{x^2}{256}+\underset{x\to 0}{o}(x^3)$

3) What is the **first-order** Taylor expansion of f about 0 ?

At order one : $f(x)=\frac{1}{16}+\underset{x\to 0}{o}(x)$

4) Deduce from 2) the value of $\lim\limits_{x\to 0}\dfrac{f(x)-\frac{1}{16}}{x}$ and the value of $\lim\limits_{x\to 0}\dfrac{f(x)-\frac{1}{16}}{x^2}$.

$\frac{f(x)-\frac{1}{16}}{x}=\frac{\frac{x^2}{256}+\underset{x\to 0}{o}(x^3)}{x}=\frac{x}{256}+\underset{x\to 0}{o}(x^2)\underset{x\to 0}{\longrightarrow}0$, and

$\frac{f(x)-\frac{1}{16}}{x^2}=\frac{1}{256}+\underset{x\to 0}{o}(x)\underset{x\to 0}{\longrightarrow}\frac{1}{256}$

EXERCISE 2: Let $f(x) = e^{\left(\frac{1}{1+x}-1\right)}$ and $g(x) = \dfrac{1}{1+3x^2}$ be two functions taken on $I=\mathbb{R}\setminus\{-1\}$.

1) Recall the second-order Taylor expansion of $x \mapsto \dfrac{1}{1+x}$ and of $x \mapsto e^x$ about 0.

$$\frac{1}{1+x} \underset{x\to 0}{=} 1 - x + x^2 + o(x^2)$$

$$e^x \underset{x\to 0}{=} 1 + x + \frac{x^2}{2} + o(x^2)$$

2) Deduce from 1) the second-order Taylor expansion of f about 0.

$e^{\frac{1}{1+x}-1}$ is of the form e^y where $y \underset{x\to 0}{\to} 0$ ($y = \frac{1}{1+x} - 1$). We have:

$$e^{\frac{1}{1+x}-1} = e^{1 - x + x^2 + \underset{x\to 0}{o(x^2)} - 1} = e^{-x + x^2 + \underset{x\to 0}{o(x^2)}}$$

$$= 1 + \left(-x + x^2\right) + \frac{\left(-x + x^2\right)^2}{2} + \underset{x\to 0}{o(x^2)}$$

$$= 1 - x + x^2 + \frac{x^2}{2} + \underset{x\to 0}{o(x^2)} = 1 - x + \frac{3}{2}x^2 + \underset{x\to 0}{o(x^2)}$$

3) Deduce from 1) and 2) the second-order Taylor expansion of the product $x \mapsto f(x)g(x)$ about 0.

$\dfrac{1}{1+3x^2} \underset{x\to 0}{=} 1 - 3x^2 + o(x^2)$ So $f(x)g(x) = \left(1 - x + \frac{3}{2}x^2\right)\left(1 - 3x^2\right) + \underset{x\to 0}{o(x^2)}$

$$= 1 - x + \frac{3}{2}x^2 - 3x^2 + \underset{x\to 0}{o(x^2)}$$

$$= 1 - x - \frac{3}{2}x^2 + \underset{x\to 0}{o(x^2)}$$

EXERCISE 3: Consider the hyperbolic functions $x \mapsto \cosh(x) = \dfrac{e^x + e^{-x}}{2}$ and $x \mapsto \sinh(x) = \dfrac{e^x - e^{-x}}{2}$ on $\mathbb{R}$.

1) Give the third-order Taylor expansions of cosh and sinh about 0.

$$\cosh(x) = \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left(\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \underset{x\to 0}{o(x^3)}\right) + \left(1 - x + \frac{(-x)^2}{2} + \frac{(-x)^3}{3!} + \underset{x\to 0}{o(x^3)}\right)\right) = 1 + \frac{x^2}{2} + \underset{x\to 0}{o(x^2)}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{6} + \underset{x\to 0}{o(x^3)}$$

2) Give the third-order Taylor expansion of the product $x \mapsto \cosh(x)\sinh(x)$ about 0.

From 1) we get : $\cosh(x)\sinh(x) = \left(1 + \frac{x^2}{2} + \underset{x\to 0}{o}(x^3)\right)\left(x + \frac{x^3}{6} + \underset{x\to 0}{o}(x^3)\right)$

$= x + \frac{x^3}{6} + \frac{x^3}{2} + \underset{x\to 0}{o}(x^3) = x + \frac{2x^3}{3} + \underset{x\to 0}{o}(x^3)$

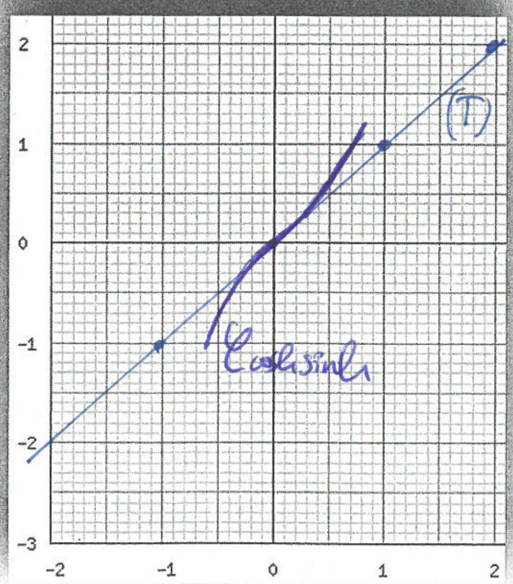
3) Give an equation of the Tangent line (T) of the graph of $x \mapsto \cosh(x)\sinh(x)$ at x=0.

From 2), $\cosh(0)\sinh(0) = 0$ and $(\cosh\sinh)'(0) = 1$ So : $(T) : y = x$ is the tangent line of cosh sinh at $x = 0$.

4) What is the position of (T) with respect to the graph of $x \mapsto \cosh(x)\sinh(x)$ near 0 ? Justify your answer.

We have $\cosh(x)\sinh(x) - x = \frac{2}{3}x^3 + \underset{x\to 0}{o}(x^3)$, so for any $x < 0$ (x close to 0) the graph $\mathcal{C}_{\cosh\sinh}$ is below its tangent line (T). And for any $x > 0$ (x close to 0), $\mathcal{C}_{\cosh\sinh}$ is above (T).

5) Give a local graphical representation of the graph of $x \mapsto \cosh(x)\sinh(x)$ near 0.



EXERCISE 4: Using Taylor expansions, calculate $\lim_{x\to 0} \frac{e^{1-5x} - e}{x}$ and $\lim_{x\to 0} \frac{\ln(3+5x) - \ln(3)}{x}$

$\frac{e^{1-5x} - e}{x} = \frac{e^1[1 - 5x + \underset{x\to 0}{o}(x)] - e}{x} = -5e + \underset{x\to 0}{o}(1) \underset{x\to 0}{\longrightarrow} -5e$,

$\frac{\ln(3+5x) - \ln(3)}{x} = \frac{\ln(3) + \ln(1 + \frac{5}{3}x) - \ln(3)}{x} = \frac{\frac{5}{3}x + \underset{x\to 0}{o}(x)}{x} = \frac{5}{3} + \underset{x\to 0}{o}(1) \underset{x\to 0}{\longrightarrow} \frac{5}{3}$

EXERCICE 5: Find an antiderivative of the following function f on I:

1) $f(x) = x^2 + x + \frac{1}{x} - \frac{1}{x^2}$, for any x in I =]0, + ∞[.

The function $F : \mathbb{R}_+^* \to \mathbb{R}$ defined by:
$F(x) = \frac{x^3}{3} + \frac{x^2}{2} + \ln(x) + \frac{1}{x}$ is an antiderivative of f on $\mathbb{R}_+^*$.

2) $f(x) = \frac{x^2}{8x^3 + 4}$, for any x in I =]0, + ∞[.

The function F defined on $\mathbb{R}_+^*$ by: $F(x) = \frac{1}{24}\ln(8x^3+4)$ is an antiderivative of f on $\mathbb{R}_+^*$.

EXERCISE 6: Let f be the function defined on I=]− 3, + ∞[by the relation: $f(x) = \ln(x + 3)$

1) Find the real number a such that $F(x) = (x + a)\ln(x + a) - x$ be an antiderivative of f on I.

Given $a \in \mathbb{R}$, F is antiderivative of f on I $\Rightarrow$ $\ln(x+a) = f(x) = \ln(x+3)$
thus $a = 3$.

2) Deduce from 1) the antiderivative of f taking the value 0 at x = −2

The antiderivative of f taking the value 0 at $x=-2$ is the function $x \mapsto F(x) + c$ where c satisfies: $F(-2) + c = 0$
i.e, $\ln(1) - (-2) + c = 0$ i.e $c = -2$. thus it is $x \mapsto (x+3)\ln(x+3) - x - 2$

EXERCISE 7: Let f be the function defined on I= $]\frac{1}{2}, + \infty[$ by the relation: $f(x) = \frac{3}{4x^2 + 4x - 3}$

1) Find the real numbers a and b such that we have: $f(x) = \frac{a}{2x-1} + \frac{b}{2x+3}$, for any x in I.

Given a, b in $\mathbb{R}$: $\frac{a}{2x-1} + \frac{b}{2x+3} = \frac{(a+b)2x + 3a - b}{4x^2+4x-3}$ and so $f(x) = \frac{a}{2x-1} + \frac{b}{2x+3}$

$\Leftrightarrow \begin{cases} (a+b)2x = 0 \\ 3a - b = 3 \end{cases} \Leftrightarrow \begin{cases} b = -a \\ a = \frac{3}{4} \end{cases}$

2) Deduce from 1) the antiderivative of f taking the value 0 at x = 2

From 1), an antiderivative of f is of the form $F : x \mapsto \frac{3}{8}\ln(2x-1) - \frac{3}{8}\ln(2x+3) + c$
where c is a constant. So $F(2) = 0 \Leftrightarrow \frac{3}{8}\ln(3) - \frac{3}{8}\ln(7) + c = 0 \Leftrightarrow c = \frac{3}{8}\ln\left(\frac{7}{3}\right)$
the desired antiderivative is thus $x \mapsto \frac{3}{8}\ln(2x-1) - \frac{3}{8}\ln(2x+3) + \frac{3}{8}\ln\left(\frac{7}{3}\right)$.