FDEG FACULTÉ DE DROIT ÉCONOMIE & GESTION UNIVERSITÉ D'ORLÉANS	**Licence 2 EG - Gestion (voie euro) - semestre 1, 2021-2022** *MATHEMATICS* **Midterm Exam 2 (1h30)** **Documents and calculators are not allowed**
SURNAME :	Forename :

All your calculations must be sufficiently justified. It might not be necessary to do all the exercises to get 20/20

EXERCISE 1: Calculate the following integrals, by **determining an antiderivative** of the functions considered:

1) $A = \int_0^{\sqrt{3}/2} \frac{2}{4x^2+3} dx$

$A = \frac{2}{3}\int_0^{\frac{\sqrt{3}}{2}} \frac{1}{\frac{4}{3}x^2+1} dx = \frac{2}{3}\left[\frac{\sqrt{3}}{2}\text{Arctan}\left(\frac{2}{\sqrt{3}}x\right)\right]_0^{\frac{\sqrt{3}}{2}}$

$= \frac{1}{\sqrt{3}}\text{Arctan}(1) - \frac{1}{\sqrt{3}}\text{Arctan}(0)$

$A = \frac{\pi}{4\sqrt{3}} - 0 = \frac{\sqrt{3}\pi}{12}$

2) $B = -2\int_1^5 \frac{1}{(1+2x)^2} e^{\frac{1}{1+2x}} dx$

$B = \int_1^5 \frac{-2}{(1+2x)^2} e^{\frac{1}{1+2x}} dx = \left[e^{\frac{1}{1+2x}}\right]_1^5 = e^{\frac{1}{11}} - e^{\frac{1}{3}}$

3) $C = \int_{1/2}^{3/4} \frac{2}{(4x-1)^4} dx$

$C = 2\int_{\frac{1}{2}}^{3/4} (4x-1)^{-4} dx = 2\left[-\frac{1}{12}(4x-1)^{-3}\right]_{1/2}^{3/4}$

$= -\frac{1}{6}\left(\frac{1}{8} - 1\right) = \frac{7}{48}$

EXERCISE 2: Find the values b in $\mathbb{R}$ such that $D = \int_2^b \frac{x}{\sqrt{3x^2+4}} dx$ be equal to $\frac{10}{3}$.

For any $b \geq 2$, we have: $D = \left[\frac{\sqrt{3x^2+4}}{3}\right]_2^b = \frac{1}{3}\left(\sqrt{3b^2+4} - \sqrt{16}\right)$

$D = \frac{1}{3}\left(\sqrt{3b^2+4} - 4\right)$ and thus $D = \frac{10}{3} \Leftrightarrow \sqrt{3b^2+4} = 14$

$\Leftrightarrow 3b^2+4 = 196$ (since $\sqrt{\ }$ is ≥ 0)

$\Leftrightarrow 3b^2 = 192$

$\Leftrightarrow b = 8$ or -8

EXERCISE 3: 1) Find the reals a and b such that: $3x^2 - 3x + \frac{3}{2} = a[(x+b)^2 + \frac{1}{4}]$

We have: $3x^2-3x+\frac{3}{2} = 3\left[\left(x-\frac{1}{2}\right)^2 - \frac{1}{4} + \frac{3}{6}\right] = 3\left[\left(x-\frac{1}{2}\right)^2+\frac{1}{4}\right]$ so $a=3$ and $b=-\frac{1}{2}$

Other method: $a\left[(x+b)^2+\frac{1}{4}\right] = ax^2+2abx+ab^2+\frac{a}{4}$ which gives by identification of coefficients: $\begin{cases} a=3\\ 2ab=-3 \\ ab^2+\frac{a}{4}=\frac{3}{2}\end{cases} \iff \begin{cases}a=3\\ b=-\frac{1}{2}\\ b^2=\frac{1}{4}\end{cases} \iff \begin{cases}a=3\\ b=-\frac{1}{2}\end{cases}$

2) Deduce from 1) the value of $A = \int_0^{\frac{1}{2}} \frac{1}{(3x^2-3x+\frac{3}{2})}dx$.

$$A = \int_0^{\frac{1}{2}} \frac{1}{a\left[(x+b)^2+\frac{1}{4}\right]}dx = \int_0^{1/2} \frac{4}{a\left[4(x+b)^2+1\right]}dx = \frac{4}{a}\int_0^{\frac{1}{2}} \frac{dx}{[2(x+b)]^2+1}$$

$$= \frac{4}{a}\left[\frac{\mathrm{Arctan}\,(2(x+b))}{2}\right]_0^{1/2} = \frac{2}{a}\left(\mathrm{Arctan}(0) - \mathrm{Arctan}(-1)\right)$$

$$\Rightarrow \boxed{A = \frac{2}{3}\left(\frac{\pi}{4}\right) = \frac{\pi}{6}.}$$

3) Give the value of $B = \int_0^{\frac{1}{2}} \frac{6x-3}{(3x^2-3x+\frac{3}{2})}dx$? Hence, deduce that $C = \int_0^{\frac{1}{2}} \frac{x}{(3x^2-3x+\frac{3}{2})}dx = -\ln(2)$.

$$B = \left[\ln\left(3x^2-3x+\frac{3}{2}\right)\right]_0^{1/2} = \ln\left(\frac{3}{4}\right) - \ln\left(\frac{3}{2}\right) = -\ln(2)$$

$$C = \frac{1}{6}\int_0^{\frac{1}{2}} \frac{6x}{\left(3x^2-3x+\frac{3}{2}\right)}dx = \frac{1}{6}\int_0^{1/2} \frac{(6x-3+3)}{(3x^2-3x+\frac{3}{2})}dx = \frac{1}{6}B + \frac{1}{2}A$$

$$= -\frac{1}{6}\ln(2) + \frac{\pi}{12}$$

4) Find a_1, b_1 in $\mathbb{R}$ such that $\overbrace{\frac{a_1}{1-x} + \frac{b_1 x}{(3x^2-3x+\frac{3}{2})}}^{=\frac{1}{-3x^3+6x^2-\frac{9}{2}x+\frac{3}{2}}}$. Deduce the value of $I = \int_0^{\frac{1}{2}} \frac{(-1)}{(3x^3-6x^2+\frac{9x}{2}-\frac{3}{2})}dx$

Given a_1, b_1 in $\mathbb{R}$, we have: $\frac{a_1}{1-x} + \frac{b_1 x}{3x^2-3x+\frac{3}{2}} = \frac{(3a_1-b_1)x^2-(3a_1-b_1)x+\frac{3}{2}a_1}{-3x^3+6x^2-\frac{9}{2}x+3/2}$

which gives: $\begin{cases} 3a_1-b_1=0 \\ 3a_1-b_1=0 \\ \frac{3}{2}a_1=1 \end{cases}$ i.e $\begin{cases} a_1=\frac{2}{3} \\ b_1=2 \end{cases}$. Hence $I = a_1[-\ln(1-x)]_0^{1/2} + b_1 C$

i.e $I = a_1\ln(2) + b_1 C$

i.e $I = \frac{2}{3}\ln(2) + 2C = \frac{2}{3}\ln(2) - \frac{1}{3}\ln(2) + \frac{\pi}{6}$

$= \frac{1}{3}\ln(2) + \frac{\pi}{6}$

EXERCISE 4: Using **integrations by parts** (among other tools), calculate the following definite integrals:

1) $A = \int_{-1}^{3} \ln(t+2)\,dt$

$A = \int_{-1}^{3} \underbrace{1}_{u'} \times \underbrace{\ln(t+2)}_{v}\,dt = \left[t\ln(t+2)\right]_{-1}^{3} - \int_{-1}^{3} \frac{t}{t+2}\,dt$

$= 3\ln(5) - \int_{-1}^{3} \frac{t+2-2}{t+2}\,dt$

$= 3\ln(5) - \left[t - 2\ln(t+2)\right]_{-1}^{3}$

$A = 3\ln(5) - (3 - 2\ln(5) + 1) = 5\ln(5) - 4$

4) $C = \int_{0}^{1} 6x^3 e^{3x^2}\,dx$

$C = \int_0^1 \underbrace{x^2}_{u} \times \underbrace{(6x)e^{3x^2}}_{v'}\,dx = \left[x^2 e^{3x^2}\right]_0^1 - 2\int_0^1 x\,e^{3x^2}\,dx$

$= e^3 - \frac{2}{6}\left[e^{3x^2}\right]_0^1 = e^3 - \frac{1}{3}(e^3 - 1) = \frac{2}{3}e^3 + \frac{1}{3}$

EXERCISE 5: Let f: I =]10, + ∞[→ ℝ be defined by $f(x) = 5x(\frac{1}{2}x - 5)^{-\frac{1}{3}}$

1) Given $x \geq 12$, calculate the definite integral $\int_{12}^{x} f(t)\,dt$. (Hint: use a substitution or an integration by parts)

Let $x \geq 12$, $\int_{12}^{x} f(t)\,dt = \int_{12}^{x} \underbrace{5t}_{u} \underbrace{(\tfrac{1}{2}t-5)^{-\frac{1}{3}}}_{v'}\,dt = \left[\tfrac{3}{2}\times 2\times 5t\,(\tfrac{1}{2}t-5)^{\frac{2}{3}}\right]_{12}^{x} - \int_{12}^{x} 15(\tfrac{1}{2}t-5)^{\frac{2}{3}}$

$= 15x(\tfrac{1}{2}x-5)^{\frac{2}{3}} - 15\times 12 - 15\left[\tfrac{3\times 2}{5}(\tfrac{1}{2}t-5)^{\frac{5}{3}}\right]_{12}^{x}$

$= 15x(\tfrac{1}{2}x-5)^{2/3} - 180 - 18\left((\tfrac{1}{2}x-5)^{5/3} - 1\right) = 15(\tfrac{1}{2}x-5)^{\frac{2}{3}}x - 162 - 18(\tfrac{1}{2}x$

2) Deduce from 1) the antiderivative that vanishes at x = **26**

Setting $F(x) = \int_{12}^{x} f(t)\,dt$, then $F'(x) = f(x)$ i.e F is an antiderivat. of f. The antiderivative that vanishes at $x = 26$ is $x \mapsto F(x) + C$ where $F(26) + C = 0$ i.e $15\times 4\times 23 - 135 + C = 0$ i.e $C = 1245$

EXERCISE 6: Using **substitutions** (among other tools), calculate the following integrals:

1) $A = \displaystyle\int_{e^{-5}}^{e} \frac{\ln(t)+3}{t\sqrt{4-\ln(t)}}\,dt.$ (Hint: set $U = 4 - \ln(t)$)

$$\begin{cases} U = 4 - \ln(t) \\ du = -\frac{1}{t}\,dt \end{cases}, \quad A = \int_9^3 -\frac{4-u+3}{\sqrt{u}}\,du = -\int_9^3 \frac{7-u}{\sqrt{u}}\,du$$

$$= -7\left[2\sqrt{u}\right]_9^3 + 7\left[\tfrac{2}{3}u^{3/2}\right]_9^3 = -14(\sqrt{3}-3) + 7(2\sqrt{3}-18)$$

$$A = 52 - 126 = -74$$

2) $B = \displaystyle\int_4^9 \frac{1}{\sqrt{x}(2\sqrt{x}-3)}\,dx.$

$$\text{Set: } \begin{cases} u = \sqrt{x} \\ du = \frac{1}{2u}\,dx \end{cases}, \quad B = \int_2^3 \frac{2\cancel{u}\,du}{\cancel{u}(2u-3)} = \left[\ln(2u-3)\right]_2^3$$

$$B = \ln(3)$$

3) $C = \displaystyle\int_0^{\ln(2)} \sqrt{e^x - 1}\,dx.$ (Bonus)

$$\text{Set: } \begin{cases} U = \sqrt{e^x - 1} \\ du = \frac{e^x}{2U}\,dx = \frac{(U^2+1)}{2u}\,dx \end{cases}$$

$$C = \int_0^1 \frac{2U^2}{U^2+1}\,du = 2\int_0^1 \frac{U^2+1-1}{U^2+1}\,du = 2\left[u - \mathrm{Arctan}(u)\right]_0^1$$

$$C = 2\left(1 - \frac{\pi}{4}\right)$$