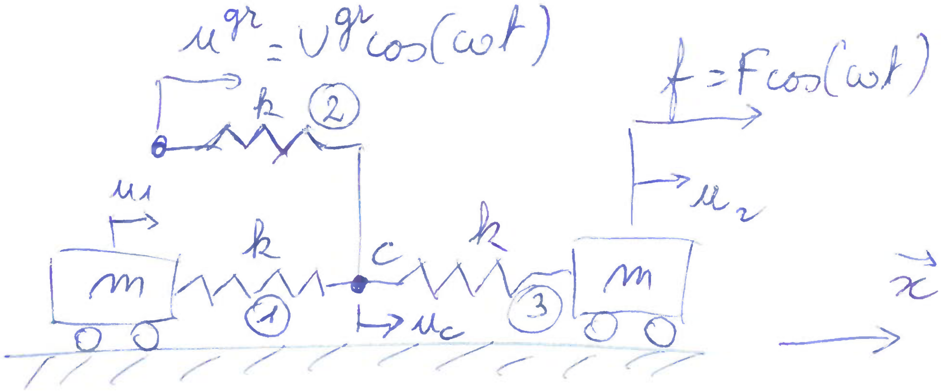
Correction DS2

Exercice 1



1) * On isole le point C (sans masse)

BAME sur l'axe $\vec{x}$:

- ressort 1 : $F_1 = -k(u_c - u_1)$
- ressort 2 : $F_2 = -k(u_c - u^{gr})$
- ressort 3 : $F_3 = k(u_2 - u_c)$

PFD : $-k(u_c - u_1) - k(u_c - u^{gr}) + k(u_2 - u_c) = 0$

$$\Rightarrow 3u_c - u_1 - u_2 - u^{gr} = 0$$

$$\Rightarrow \boxed{u_c = \frac{1}{3}(u_1 + u_2 + u^{gr})}$$

* On isole la masse de gauche

BAME sur l'axe $\vec{x}$:

- ressort 1 : $F_1 = k(u_c - u_1)$

PFD : $k(u_c - u_1) = m\ddot{u}_1$

$$\Rightarrow m\ddot{u}_1 + k(u_1 - u_c) = 0$$

$$\Rightarrow m\ddot{u}_1 + ku_1 - \frac{k}{3}(u_1 + u_2 + u^{gr}) = 0$$

$$\Rightarrow \boxed{m\ddot{u}_1 + \frac{2k}{3}u_1 - \frac{k}{3}u_2 = \frac{k}{3}u^{gr}} \quad (1)$$

* On isole la masse de droite :

BAME sur l'axe $\vec{x}$:

- ressort 3 : $-k(u_2 - u_c)$
- effort f

PFD : $-k(u_2 - u_c) + f = m\ddot{u}_2$

$$\Rightarrow m\ddot{u}_2 + k u_2 - \frac{k}{3}(u_1 + u_2 + u^{gr}) = f$$

$$\Rightarrow \boxed{m\ddot{u}_2 + \frac{2k}{3}u_2 - \frac{k}{3}u_1 = f + \frac{k}{3}u^{gr}} \quad (2)$$

2) On cherche une solution stationnaire (solution particulière) de la forme :

$$\begin{cases} u_1 = U_1 \cos(\omega t) \\ u_2 = U_2 \cos(\omega t) \end{cases}$$

D'où :

$$(1) \Rightarrow -m\omega^2 U_1 + \frac{2k}{3}U_1 - \frac{k}{3}U_2 = \frac{k}{3}U^{gr}$$

$$\Rightarrow U_1\left(\frac{2}{3} - \frac{m}{k}\omega^2\right) - \frac{1}{3}U_2 = \frac{1}{3}U^{gr}$$

$$\Rightarrow U_1\left[\frac{2}{3} - \left(\frac{\omega}{\omega_0}\right)^2\right] - \frac{1}{3}U_2 = \frac{1}{3}U^{gr} \quad (3)$$

$$(2) \Rightarrow -m\omega^2 U_2 + \frac{2k}{3}U_2 - \frac{k}{3}U_1 = F + \frac{k}{3}U^{gr}$$

$$\Rightarrow U_2\left[\frac{2}{3} - \left(\frac{\omega}{\omega_0}\right)^2\right] - \frac{1}{3}U_1 = \frac{F}{k} + \frac{1}{3}U^{gr} \quad (4)$$

Résolution :

$$(3) \Rightarrow \left[2-3\left(\frac{\omega}{\omega_0}\right)^2\right] U_1 - U_2 = U^{gr}$$

$$\Rightarrow U_2 = \left[2-3\left(\frac{\omega}{\omega_0}\right)^2\right] U_1 - U^{gr}$$

$$(4) \Rightarrow \left[2-3\left(\frac{\omega}{\omega_0}\right)^2\right] U_2 - U_1 = \frac{3F}{k} + U^{gr}$$

$$\Rightarrow \left[2-3\left(\frac{\omega}{\omega_0}\right)^2\right]\left[\left(2-3\left(\frac{\omega}{\omega_0}\right)^2\right) U_1 - U^{gr}\right] - U_1 = \frac{3F}{k} + U^{gr}$$

$$\Rightarrow \left[2-3\left(\frac{\omega}{\omega_0}\right)^2\right]^2 U_1 - \left[2-3\left(\frac{\omega}{\omega_0}\right)^2\right] U^{gr} - U_1 = \frac{3F}{k} + U^{gr}$$

$$\Rightarrow \left(\left[2-3\left(\frac{\omega}{\omega_0}\right)^2\right]^2 - 1\right) U_1 = \frac{3F}{k} + \left(\left[2-3\left(\frac{\omega}{\omega_0}\right)^2\right] + 1\right) U^{gr}$$

$$\Rightarrow \left[3 - 12\left(\frac{\omega}{\omega_0}\right)^2 + 9\left(\frac{\omega}{\omega_0}\right)^4\right] U_1 = \frac{3F}{k} + \left[3-3\left(\frac{\omega}{\omega_0}\right)^2\right] U^{gr}$$

$$\Rightarrow \boxed{U_1 = \frac{F/k + \left[1-\left(\frac{\omega}{\omega_0}\right)^2\right] U^{gr}}{1-4\left(\frac{\omega}{\omega_0}\right)^2 + 3\left(\frac{\omega}{\omega_0}\right)^4}}$$

Puis :

$$U_2 = \left[2-3\left(\frac{\omega}{\omega_0}\right)^2\right] U_1 - U^{gr}$$

$$= \left[2-3\left(\frac{\omega}{\omega_0}\right)^2\right]\left(\frac{F/k + \left[1-\left(\frac{\omega}{\omega_0}\right)^2\right] U^{gr}}{1-4\left(\frac{\omega}{\omega_0}\right)^2 + 3\left(\frac{\omega}{\omega_0}\right)^4}\right) - U^{gr}$$

$$= \frac{\left[2-3\left(\frac{\omega}{\omega_0}\right)^2\right]\frac{F}{k} + \left[2-3\left(\frac{\omega}{\omega_0}\right)^2\right]\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right] U^{gr} - \left[1-4\left(\frac{\omega}{\omega_0}\right)^2 + 3\left(\frac{\omega}{\omega_0}\right)^4\right] U^{gr}}{1-4\left(\frac{\omega}{\omega_0}\right)^2 + 3\left(\frac{\omega}{\omega_0}\right)^4}$$

Donc :

$$V_2 = \frac{\left[2-3\left(\frac{\omega}{\omega_0}\right)^2\right]\frac{F}{k} + \left[2-5\left(\frac{\omega}{\omega_0}\right)^2+3\left(\frac{\omega}{\omega_0}\right)^4-1+4\left(\frac{\omega}{\omega_0}\right)^2-3\left(\frac{\omega}{\omega_0}\right)^4\right]U^{gr}}{1-4\left(\frac{\omega}{\omega_0}\right)^2+3\left(\frac{\omega}{\omega_0}\right)^4}$$

$$\Rightarrow \boxed{V_2 = \frac{\left[2-3\left(\frac{\omega}{\omega_0}\right)^2\right]\frac{F}{k} + \left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]U^{gr}}{1-4\left(\frac{\omega}{\omega_0}\right)^2+3\left(\frac{\omega}{\omega_0}\right)^4}}$$

3) Il faut que $\det([K]-\omega^2[M])=0$

avec $[M]=\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$ et $[K]=\begin{pmatrix} \frac{2k}{3} & -\frac{k}{3} \\ -\frac{k}{3} & \frac{2k}{3} \end{pmatrix}$

$$\Rightarrow \begin{vmatrix} \frac{2k}{3}-m\omega^2 & -k/3 \\ -k/3 & \frac{2k}{3}-m\omega^2 \end{vmatrix} = 0$$

$$\Rightarrow \left(\frac{2k}{3}-m\omega^2\right)^2 - \frac{k^2}{9} = 0$$

$$\Rightarrow \frac{4k^2}{9} - \frac{4km}{3}\omega^2 + m^2\omega^4 - \frac{k^2}{9} = 0$$

$$\Rightarrow m^2\omega^4 - \frac{4km}{3}\omega^2 + \frac{k^2}{3} = 0$$

$$\Rightarrow 3\frac{m^2}{k^2}\omega^4 - 4\frac{m}{k}\omega^2 + 1 = 0$$

$$\Rightarrow \frac{3}{\omega_0^4}\omega^4 - \frac{4}{\omega_0^2}\omega^2 + 1 = 0 \Rightarrow 3\omega^4 - 4\omega_0^2\omega^2 + \omega_0^4 = 0$$

Resolution :

$$\Delta = 16\omega_0^4 - 12\omega_0^4 = 4\omega_0^4$$

D'où :

$$\begin{cases} \omega_1^2 = \dfrac{4\omega_0^2 + 2\omega_0^2}{6} = \omega_0^2 \\ \omega_2^2 = \dfrac{4\omega_0^2 - 2\omega_0^2}{6} = \dfrac{1}{3}\omega_0^2 \end{cases}$$

Donc : $\begin{cases} \omega_1 = \omega_0 \\ \omega_2 = \dfrac{1}{\sqrt{3}}\omega_0 \end{cases}$ avec $\omega_0 = \sqrt{\dfrac{k}{m}}$

Exercice 2

1) * Masses modales : $m_i = [X]_i^T [M] [X]_i$

D'où :

$$m_1 = [X]_1^T [M] [X]_1 = \begin{pmatrix} 1/2 & 1 \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & 2m \end{pmatrix} \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 1 \end{pmatrix} \begin{pmatrix} m/2 \\ 2m \end{pmatrix} = \frac{m}{4} + 2m = \frac{9}{4}m$$

$$m_2 = [X]_2^T [M] [X]_2 = \begin{pmatrix} -4 & 1 \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & 2m \end{pmatrix} \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 1 \end{pmatrix} \begin{pmatrix} -4m \\ 2m \end{pmatrix} = 16m + 2m = 18m$$

* Rigidités modales : $k_i = [X]_i^T [K] [X]_i$

$$k_1 = [X]_1^T [K] [X]_1 = \begin{pmatrix} 1/2 & 1 \end{pmatrix} \begin{pmatrix} 5k & -2k \\ -2k & 3k \end{pmatrix} \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$$

$$\Rightarrow k_1 = \begin{pmatrix} 1/2 & 1 \end{pmatrix} \begin{pmatrix} \frac{5}{2}k - 2k \\ -k + 3k \end{pmatrix} = \begin{pmatrix} 1/2 & 1 \end{pmatrix} \begin{pmatrix} k/2 \\ 2k \end{pmatrix}$$

$$= \frac{k}{4} + 2k = \frac{9}{4} k$$

$$k_2 = [X]_2^T [K] [X]_2 = \begin{pmatrix} -4 & 1 \end{pmatrix} \begin{pmatrix} 5k & -2k \\ -2k & 3k \end{pmatrix} \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 1 \end{pmatrix} \begin{pmatrix} -22k \\ 11k \end{pmatrix} = 88k + 11k = 99k$$

2) Pulsations propres :

$$\omega_1 = \frac{k_1}{m_1} = \frac{9/4\, k}{9/4\, m} = \frac{k}{m}$$

$$\omega_2 = \frac{k_2}{m_2} = \frac{99k}{18m} = \frac{11}{2} \frac{k}{m}$$

3) Amplitudes modales solution de $\ddot{\phi}_i + \omega_i^2 \phi_i = 0$

$$\Rightarrow \phi_i = A_i \cos(\omega_i t) + B_i \sin(\omega_i t)$$

$$\Rightarrow \left| \begin{array}{l} \phi_1 = A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t) \\ \phi_2 = A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t) \end{array} \right.$$

4) On en déduit :

$$[u] = \phi_1 [X]_1 + \phi_2 [X]_2 \qquad \text{avec} \quad [u] = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\Rightarrow \left| \begin{array}{l} u_1 = \frac{1}{2} A_1 \cos(\omega_1 t) + \frac{1}{2} B_1 \sin(\omega_1 t) - 4A_2 \cos(\omega_2 t) - 4B_2 \sin(\omega_2 t) \\ u_2 = A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t) + A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t) \end{array} \right.$$